UJAT
Universidad Juárez Autónoma de Tabasco
"Estudio en la duda. Acción en la fe"

Diseño educativo para la enseñanza de la cinemática de mecanismo

-Perera-Cortez et al.

JOBS
DACB • UJAT
Journal of Basic Sciences

Volúmen 12
Número 34
mayo-agosto 2026

ISSN:2448-4997
https://revistajobs.ujat.mx

Journal of Basic Sciences

En el número 34 del Journal of Basic Sciences se presentan ocho contribuciones que reflejan el carácter multidisciplinario de la revista, al abordar temas relacionados con ingeniería y ciencias de materiales, enseñanza de las ciencias, farmacéutica, física y matemáticas. Desde sus respectivas áreas, los trabajos emplean herramientas científicas para ampliar el conocimiento, fortalecer los procesos de enseñanza-aprendizaje y contribuir a la solución de problemáticas de interés social, tecnológico y académico.

El número inicia con el artículo de López-Zapata et al., en el que se evalúa la corrosión atmosférica del acero al carbono y cobre en el municipio de Cunduacán, Tabasco, considerando la presencia de contaminantes atmosféricos asociados al entorno. Los resultados destacan la importancia de fortalecer las estrategias de control de emisiones y protección de materiales empleados en infraestructura industrial y residencial, particularmente en regiones con actividad petrolera.

Posteriormente, se presentan dos contribuciones relacionadas con la enseñanza de las ciencias. Perera Cortez et al. desarrollan y validan un prototipo didáctico de un mecanismo biela-manivela con corredera para la enseñanza de la cinemática en estudiantes de ingeniería. El dispositivo integra teoría, simulación y mediciones experimentales, favoreciendo una comprensión más interactiva de los conceptos estudiados. Por su parte, López-de Dios et al. analizan el uso y valoración de herramientas informáticas por profesores de la Licenciatura en Química de la Universidad Juárez Autónoma de Tabasco durante la transición hacia la enseñanza virtual ocasionada por la pandemia de COVID-19. Sus resultados resaltan la necesidad de fortalecer no solamente las competencias técnicas digitales, sino también su adecuada integración pedagógica en la enseñanza de disciplinas experimentales.

En el área farmacéutica, Díaz-Martínez et al. comparan la ciclofosfamida y la bendamustina para el tratamiento del cáncer desde una perspectiva farmacoterapéutica y farmacoeconómica. El estudio considera eficacia, costos, reacciones adversas y disponibilidad de los medicamentos, proporcionando elementos para apoyar la toma de decisiones clínicas. Los resultados muestran ventajas económicas para la ciclofosfamida, mientras que la bendamustina representa una alternativa cuando se priorizan aspectos relacionados con tolerabilidad y seguridad.

Dentro del área de física, Rendón et al. estudian la variación de la presión atmosférica con respecto a la altura mediante modelos que incorporan un gradiente lineal de temperatura en las ecuaciones de estado de los gases ideales y de van der Waals. Esta aproximación permite representar de manera más realista el comportamiento de la atmósfera terrestre y explorar posibles comportamientos en Venus, Marte y Titán.

El número incluye además tres contribuciones del área de matemáticas. Balcázar-Araiza y Ojeda presentan una generalización dentro del Cálculo Fraccionario Multivariable mediante la definición y análisis de integrales y diferenciales de Riemann-Liouville de tipo (α, k, P), estableciendo propiedades y teoremas fundamentales asociados a estos operadores. Toledo et al. desarrollan un método numérico basado en elementos finitos para resolver ecuaciones diferenciales unidimensionales que involucran la derivada conformable, ofreciendo una herramienta para la aproximación eficiente de este tipo de problemas. Finalmente, Remigio-Juárez et al. presentan una introducción a los complejos de Vietoris-Rips y a la homología persistente, incluyendo el uso de diagramas de persistencia y herramientas computacionales en Python para el análisis de conjuntos de datos, con posibles aplicaciones en diversas áreas científicas.

De este modo, el número 34 del Journal of Basic Sciences reúne investigaciones y contribuciones académicas que muestran la diversidad de enfoques de las ciencias básicas y sus aplicaciones. En conjunto, los artículos fortalecen la difusión y divulgación del conocimiento científico y buscan generar impacto en los ámbitos regional, nacional e internacional.

DIRECTORIO INSTITUCIONAL

Universidad Juárez Autónoma de Tabasco

Lic. Guillermo Narváez Osorio.
Rector

Dr. Luis Manuel Hernández Govea.
Secretario de Servicios Académicos

Dr. Wilfrido Miguel Contreras Sánchez.
Secretario de Investigación, Posgrado y Vinculación

Dr. Pablo Marín Olán. Director de Difusión,
Divulgación Científica y Tecnológica

Directorio Divisional
División Académica de Ciencias Básicas

Dra. Hermicenda Pérez Vidal.
Directora

Dr. Luis Manuel Martínez González.
Coordinador de Investigación

M.C. Abel Cortazar May.
Coordinador de Docencia

L.Q. Esmeralda León Ramos.
Coordinador de Difusión Cultural y Extensión

ISSN:2448-4997
https://revistajobs.ujat.mx

UJAT
UNIVERSIDAD JUÁREZ
AUTÓNOMA DE TABASCO
"ESTUDIO EN LA DUDA. ACCIÓN EN LA FE"

JOURNAL OF BASIC SCIENCES, Vol. 12, Núm. 34, agosto de 2026, es una publicación continua cuatrimestral, editada por la División Académica de Ciencias Básicas de la Universidad Juárez Autónoma de Tabasco, Av. Universidad S/N, Zona de la Cultura, Col. Magisterial, C.P. 86040, Villahermosa Tabasco, México. Tel. (+52) (933) 358 1500 Ext. 5040. https://revistajobs.ujat.mx. Editor Responsable de la Revista: Addy Margarita Bolivar Cimé. Reserva de derechos al uso exclusivo 04-2015-052110084000-203, ISSN: 2448-4997, ambos otorgado por el Instituto Nacional del Derecho de Autor. Universidad Juárez Autónoma de Tabasco, Av. Universidad S/N, Zona de la Cultura, Col. Magisterial, Centro, Tabasco. C.P. 86040. Fecha de última actualización, 27 de junio de 2026.

UJAT
UNIVERSIDAD JUÁREZ AUTÓNOMA DE TABASCO
"ESTUDIO EN LA DUDA. ACCIÓN EN LA FE"

CONTENIDO

ISSN:2448-4997
https://revistajobs.ujat.mx

Vol. 12 (34), p. 101–120, mayo–agosto 2026, e-ISSN: 2448 4997
JOURNAL OF BASIC SCIENCES

Finite element method for a one-dimensional conformable differential equation with Dirichlet boundary condition
Método de elementos finitos para una ecuación diferencial conformable unidimensional con condiciones de frontera de Dirichlet

Toledo, Porfirio[1], **Hernández-Linares, Carlos Alberto**[1,*], **Avendaño-Garrido, Martha Lorena**[1]
[1] *Facultad de Matemáticas, Universidad Veracruzana, Xalapa, Ver., México.*
[*] *carlhernandez@uv.mx*

Abstract
This paper presents a numerical scheme for a one-dimensional differential equation involving the conformable derivative with Dirichlet boundary conditions, developed within the framework of the finite element method. Starting from a weak formulation based on a conformable version of the fundamental lemma of the calculus of variations, the method is implemented using piecewise-defined test functions over a partition of the domain. The locality of the conformable operator allows for a direct adaptation of standard finite element techniques. Numerical experiments are carried out for different values of the order of derivation α, the results show a smooth transition in the approximated solutions as α approaches one, consistently recovering the classical case.

Keywords: *Non-integer order calculus, Conformable derivative, Calculus of variations, Finite element method, Conformable differential equations.*

Resumen
Este artículo presenta un esquema numérico para una ecuación diferencial unidimensional que involucra la derivada conformable con condiciones de frontera de Dirichlet, desarrollado dentro del marco del método de elemento finito. Partiendo de una formulación débil basada en una versión conformable del lema fundamental del cálculo de variaciones, el método se implementa utilizando funciones de prueba definidas por partes sobre una partición del dominio. La localidad del operador conformable permite una adaptación directa de las técnicas estándar de elemento finito. Se llevan a cabo experimentos numéricos para diferentes valores del orden de derivación α, los resultados muestran una transición suave en las soluciones aproximadas a medida que α se aproxima a uno, recuperando de forma consistente el caso clásico.

Palabras claves: *Cálculo de orden no entero, Derivada conformable, Cálculo de variaciones, Método de elemento finito, Ecuaciones diferenciales conformables.*

Recibido: 31 de octubre de 2025. Aceptado: 23 de mayo de 2026. Publicado: 31 de agosto de 2026.

1 Introduction

The non-integer order calculus has its origins in the 17th century. The correspondence between L'Hôpital and Leibniz, in which they discussed the meaning of a derivative of order 1/2, is well known in this context [18]. Over time, these ideas led to the development of several fractional operators, including the Riemann–Liouville, Caputo, and Grünwald–Letnikov derivatives, as well as more recent local formulations such as the conformable derivative. These operators generalize the classical concept of differentiation and have given rise to a well-established theoretical framework with numerous applications. While classical fractional derivatives are typically nonlocal and model memory and hereditary effects in dynamical systems [6,12], local operators, such as the conformable derivative, preserve key structural properties of ordinary differential calculus [1,11,22]. This structural compatibility makes them particularly suitable for integration into variational and numerical frameworks [7,9,13,21].

Sales Teodoro et al. [22] propose a classification of fractional and related operators based on both structural and functional criteria. They divide them into four classes: classical fractional derivatives, such as Caputo, Grünwald–Letnikov, and Riemann–Liouville; modified derivatives, including Canavati, Hadamard, Hilfer, Jumarie, Marchaud, and Weyl; local operators, such as Chen, conformable, deformable, Katugampola, Kolwanker, $\mathcal{M}$-operator, and $\mathcal{V}$-truncated operator; and operators with nonsingular kernels, including Atangana–Baleanu, Caputo–Fabrizio, and Yang et al.

Starting in the 1990s, formulations of local non-integer derivatives were developed to reinterpret fractionality from a perspective more compatible with classical analysis. Unlike traditional fractional derivatives, these definitions do not rely on integral operators over intervals but are constructed through pointwise limits, allowing fundamental calculus rules to be preserved. In [22], five conditions are proposed that a local operator must satisfy: linearity, equivalence with the classical derivative when the order is one, vanishing derivative for constants, validity of the product rule, and compliance with the chain rule.

Local non-integer order operators, such as the conformable derivative, have diverse and relevant applications, extending the scope of classical differential calculus to model phenomena in biology [23,25], ecology [10,17], chemistry [2,3], physics [8,20], and engineering [5,16], among other areas. Consequently, studying differential equations involving such operators is of considerable interest, as it enables the extension of classical calculus tools to more general settings. Furthermore, when analytical solutions are difficult to obtain, these operators facilitate the development of efficient numerical methods for approximating solutions.

In general, for fractional differential equations, a wide variety of numerical methods have been developed, including finite difference schemes, predictor–corrector approaches, spectral methods, and finite element formulations. Comprehensive treatments of the analysis and numerical approximation of fractional differential equations can be found in [6,14]. In the context of conformable derivatives, the development of numerical methods is more recent. Initial contributions include discretization procedures and computational treatments for conformable differential equations [24]. Subsequently, finite element techniques adapted to conformable operators have been proposed. In particular, modified finite element schemes based on cubic B–spline basis functions have been applied to conformable space–time fractional nonlinear models [7]. More recently, a conformable finite element formulation has been presented, incorporating a variational framework, the construction of suitable basis functions, and corresponding error estimates for conformable fractional partial differential equations [21]. These contributions show that the extension of Galerkin and finite element techniques to conformable fractional operators remains an active line of research.

Among the various numerical techniques available for differential equations, the finite element method stands out due to its versatility and its ability to handle complex geometries and boundary conditions. Over the years, it has evolved alongside the modeling of physical systems across various disciplines [15]. In this context, the local structure of operators such as the conformable derivative makes them particularly well suited for integration into finite element formulations. Unlike nonlocal operators, they do not require knowledge of the entire function history, which simplifies their numerical implementation. Furthermore, their ability to preserve essential properties of the classical derivative facilitates their incorporation into standard variational frameworks. For these reasons, applying the finite element method to conformable differential equations provides a coherent extension of the classical theory to fractional models with local characteristics.

The purpose of this work is to formulate and analyze a numerical scheme for differential equations involving the conformable derivative. The proposed approach employs the finite element method and adapts the fundamental lemma of the calculus of variations to the conformable setting in order to derive the corresponding weak formulation. This formulation serves as the basis for constructing the numerical scheme through domain discretization and the use of piecewise-defined test functions. The method is applied to different derivative orders, and the results are examined to analyze how the approximated solutions respond to variations in the order parameter. The local nature of the conformable operator enables its integration into the numerical framework and facilitates its extension to broader classes of problems.

This paper is organized as follows. Section 2 presents the basic definitions and main results of conformable calculus, together with the fundamental tools from conformable variational calculus. Section 3 develops the finite element formulation for a conformable differential equation and includes a numerical example. Finally, section 4 provides the conclusions.

2 Fundamentals of conformable calculus

To establish the theoretical framework required for the numerical formulation, we now review the main concepts of conformable calculus. Among the local definitions of non-integer differential operators, one of the most widely recognized is the conformable derivative introduced by Khalil et al. [11]. This operator is defined through a limit that extends the classical notion of differentiation while preserving several of its fundamental properties. The formal definition is presented below.

2.1 Basic definitions and results

Definition 2.1. *Let $f : [0, +\infty) \to \mathbb{R}$. For $t > 0$ and $\alpha \in (0, 1]$, the conformable derivative of order α is defined by*

$$T^{\alpha} f(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon}. \tag{1}$$

This operator is local and satisfies the fundamental properties of classical calculus, such as linearity, the derivative of a constant being equal to zero, and the product, quotient, and chain rules.

Subsequently, Abdeljawad [1] modified this proposal to define left and right conformable derivatives, shifting the reference point towards the endpoints of the definition interval.

Definition 2.2. *Let $f : [a, +\infty) \to \mathbb{R}$. For $t > a$ and $\alpha \in (0, 1]$, the left derivative starting from a of order α is defined by*

$$T_a^{\alpha} f(t) = \lim_{\epsilon \to 0} \frac{f(t + \epsilon (t-a)^{1-\alpha}) - f(t)}{\epsilon}. \tag{2}$$

If f is left α-derivable starting from a at every point $t > a$ and $\lim_{t \to a^+} T_a^\alpha f(t)$ exists, then the corresponding derivative at a is defined as

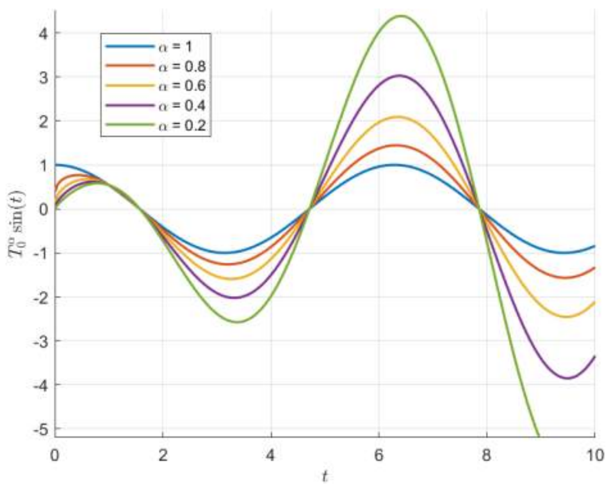
$$T_a^\alpha f(a) = \lim_{t \to a^+} T_a^\alpha f(t). \tag{3}$$

Note that if $a = 0$, we recover Khalil's definition

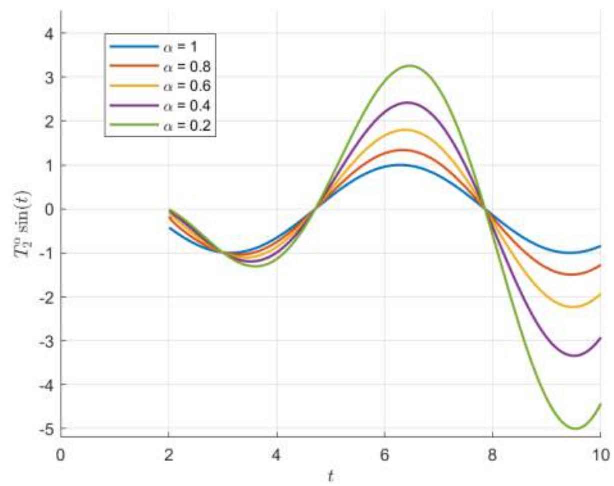
$$T^\alpha f(t) = T_0^\alpha f(t). \tag{4}$$

For a function $f : (-\infty, b] \to \mathbb{R}$, the right derivative ${}_b^\alpha T f(t)$ of order α terminating at b is defined analogously (see [1]).

To illustrate the relationship between these formulations, we consider the function $f(t) = \sin(t)$ and compute its conformable derivative for representative values of $\alpha \in (0, 1]$. The results obtained using the definition of Khalil et al. and the formulation proposed by Abdeljawad (with $a = 2$) are displayed in Figure 1(a) and Figure 1(b), respectively. The graphical comparison shows that both approaches yield curves with the same qualitative behavior. In particular, the factor $t^{1-\alpha}$ governs the amplitude variation as α decreases, while the local character of the operator is preserved in both cases.



(a) Khalil et al. definition.



(b) Abdeljawad definition.

Figure 1: Comparison of the conformable derivative of $\sin(t)$ for different values of α.

These definitions preserve the fundamental properties of classical differential calculus. From a physical point of view, the derivative is maintained as a measure of velocity, although this velocity is now modulated by a function that introduces dependency on the non-integer order. This allows for the description of trajectories of local change that are more flexible. Geometrically, this derivative represents Newton's quotient with increments advancing at a deformed scale proportional to a power of $t - a$. Furthermore, it retains the properties of the integer-order derivative definition (see [1, 11]).

Theorem 2.1. *Let $f, g : [a, +\infty) \to \mathbb{R}$ be functions that are left α-differentiable starting from a at a point $t > a$, with $\alpha \in (0, 1]$. Then, the conformable derivative T_a^α satisfies the following properties.*

1. *Linearity: For any $k_1, k_2 \in \mathbb{R}$, $T_a^\alpha(k_1 f + k_2 g)(t) = k_1 T_a^\alpha f(t) + k_2 T_a^\alpha g(t)$.*

2. *Equivalence with the ordinary derivative for order* $\alpha = 1$*:* $T_a^1 f(t) = f'(t)$.

3. *Derivative of a constant: For any constant* $k \in \mathbb{R}$, $T_a^\alpha k = 0$.

4. *Product rule:* $T_a^\alpha (fg)(t) = T_a^\alpha f(t) \cdot g(t) + f(t) \cdot T_a^\alpha g(t)$.

5. *Chain rule: If* $g(t) \neq 0$, *then* $T_a^\alpha (f \circ g)(t) = T_a^\alpha f(g(t)) \cdot T_a^\alpha g(t) \cdot g(t)^{\alpha-1}$.

The properties established in the previous result satisfy the criteria proposed in [22] for the conformable derivative to be considered a non-integer order operator. Furthermore, these properties are complemented by additional ones adopted from the classical definition, which will be useful throughout this work and are presented below for subsequent reference.

Theorem 2.2. *If* $f : [a, +\infty) \to \mathbb{R}$ *is left* α*-differentiable starting from* a *at a point* $t > a$*, with* $\alpha \in (0, 1]$*, then* f *is continuous at* t*.*

Proof. If $h \neq 0$, setting $\varepsilon = \frac{h}{(t-a)^{1-\alpha}}$, we have that $\varepsilon \to 0$ as $h \to 0$. Since f is α-differentiable at t, it follows that

$$\lim_{h\to 0} f(t+h) - f(t) = \lim_{h\to 0} (f(t+h) - f(t)) \frac{(t-a)^{1-\alpha}}{h} \frac{h}{(t-a)^{1-\alpha}} = T_a^\alpha f(t) \lim_{h\to 0} \frac{h}{(t-a)^{1-\alpha}}. \quad (5)$$

This guarantees that f is continuous at t.

□

Theorem 2.3. *If* $f : [a, +\infty) \to \mathbb{R}$ *is differentiable at* $t > a$*, then*

$$T_a^\alpha f(t) = (t-a)^{1-\alpha} f'(t), \quad (6)$$

for $\alpha \in (0, 1]$*.*

Proof. If $\varepsilon \neq 0$, setting $h = \varepsilon(t-a)^{1-\alpha}$ we have that $h \to 0$ when $\varepsilon \to 0$. Since f is differentiable at t, we have that

$$\lim_{\epsilon\to 0} \frac{f(t + \epsilon(t-a)^{1-\alpha}) - f(t)}{\epsilon} = \lim_{h\to 0} \frac{f(t+h) - f(t)}{h} (t-a)^{1-\alpha}. \quad (7)$$

From which we conclude that $T_a^\alpha f(t) = (t-a)^{1-\alpha} f'(t)$.

□

The corollaries stated below follow directly from the preceding theorem.

Corollary 2.1. *If* $f : [a, +\infty) \to \mathbb{R}$ *is twice differentiable at* $t > a$*, then* $T_a^\alpha f$ *is* α*-differentiable at* t*.*

Corollary 2.2. *If* $f : [a, +\infty) \to \mathbb{R}$ *is differentiable at* $t > a$*, then*

$$\lim_{\alpha\to 1^-} (T_a^\alpha f)(t) = f'(t), \quad (8)$$

for each $t \in (a, +\infty)$*.*

Corollary 2.3. *If* $\{f_n\}_{n=1}^\infty$*, with* $f_n : [a, \infty)$*, is a sequence of differentiable functions such that* $\{f'_n\}_{n=1}^\infty$ *converges uniformly to a function* g*. Then* $\{f_n\}_{n=1}^\infty$ *converges uniformly to a differentiable function* f*, with* $f' = g$ *and* $\{T_a^\alpha f_n\}_{n=1}^\infty$ *converges uniformly to* $T_a^\alpha f$*.*

As with the derivative, we can define the left and the right conformable integral of order α.

Definition 2.3. *Let $f : [a, +\infty) \to \mathbb{R}$. For $t > a$ and $\alpha \in (0, 1]$, the left integral of order α is defined by*

$$I_a^\alpha f(t) = \int_a^t f(s) d\alpha(s, a) = \int_a^t f(s)(s - a)^{\alpha-1} ds. \tag{9}$$

The right integral is defined analogously (see [1, 11]). It is easy to see that it is possible to calculate the left integral of order α when f is continuous. Furthermore, we have a result equivalent to the fundamental theorem of calculus.

Theorem 2.4. *If $f : [a, +\infty) \to \mathbb{R}$ is differentiable on $(a, +\infty)$, then*

$$I_a^\alpha (T_a^\alpha f)(t) = f(t) - f(a), \tag{10}$$

for $\alpha \in (0, 1]$.

Proof. According to Theorem 2.3 we have

$$I_a^\alpha (T_a^\alpha f)(t) = \int_a^t (s - a)^{1-\alpha} f'(s)(s - a)^{\alpha-1} ds = \int_a^t f'(s) ds = f(t) - f(a). \tag{11}$$

□

2.2 Piecewise defined functions

Piecewise defined functions are of particular interest in the study of differential equations, especially when numerical methods are used to approximate solutions. Many techniques, such as the finite element method, construct approximate solutions within function spaces formed by piecewise continuous or piecewise differentiable functions. This allows domains with discontinuities or abrupt changes in problem conditions to be handled efficiently. The following definitions of piecewise continuous and piecewise differentiable functions can be found in [4].

Definition 2.4. *A function $f : [a, b] \to \mathbb{R}$ is piecewise continuous on $[a, b]$ if:*

1. *The function f is bounded on $[a, b]$.*
2. *For every $t \in [a, b)$, $\lim_{s\to t^+} f(s)$ exists.*
3. *For every $t \in (a, b]$, $\lim_{s\to t^-} f(s)$ exists.*
4. *There exists a partition $\{t_0, t_1, \ldots, t_n\}$ of $[a, b]$ and f is continuous on each subinterval (t_{i-1}, t_i).*

Based on the previous definition, it is possible to define a piecewise differentiable function from a piecewise continuous one.

Definition 2.5. *A function $f : [a, b] \to \mathbb{R}$ is piecewise differentiable on $[a, b]$ if:*

1. *The function f is continuous on $[a, b]$.*
2. *There exists a piecewise continuous function g and $c \in \mathbb{R}$ such that*

$$f(t) = c + \int_a^t g(s) ds \tag{12}$$

for every $t \in [a, b]$.

For this type of function, the conclusion of Theorem 2.4 is also valid.

Theorem 2.5. *If $f : [a, +\infty) \to \mathbb{R}$ is piecewise differentiable on $[a, t]$ for every $t > a$, then*

$$I_a^\alpha (T_a^\alpha f)(t) = f(t) - f(a), \tag{13}$$

for $\alpha \in (0, 1]$.

Proof. Given $t > a$, there exists a piecewise continuous function $g : [a, t] \to \mathbb{R}$ and a constant $c \in \mathbb{R}$ such that

$$f(u) = c + \int_a^u g(s)ds, \tag{14}$$

for every $u \in [a, t]$. Let $\{t_0, t_1, \ldots, t_n\}$ be the partition of $[a, t]$ determined by g and $P = \{t_1, \ldots, t_{n-1}\}$. In this case, for every $u \in [a, t] \setminus P$ we have $f'(u) = g(u)$.

By Theorem 2.3 we have

$$\begin{aligned} I_a^\alpha (T_a^\alpha f)(t) &= \int_a^t (T_a^\alpha f)(u)(u-a)^{\alpha-1} du \\ &= \sum_{i=1}^n \int_{t_{i-1}}^{t_i} (u-a)^{1-\alpha} f'(u)(u-a)^{\alpha-1} du \\ &= \sum_{i=1}^n (f(t_i) - f(t_{i-1})) \\ &= f(t) - f(a). \end{aligned} \tag{15}$$

□

Theorem 2.6 (Integration by parts)**.** *If $f, g : [a, +\infty) \to \mathbb{R}$ are differentiable on $(a, +\infty)$ and $b > a$, then*

$$\int_a^b f(s) T_a^\alpha g(s) d\alpha(s, a) = f(b)g(b) - f(a)g(a) - \int_a^b g(s) T_a^\alpha f(s) d\alpha(s, a). \tag{16}$$

Proof. By the product rule, we have that

$$I_a^\alpha (T_a^\alpha fg)(b) = \int_a^b f(s)(T_a^\alpha g)(s) d\alpha(s, a) + \int_a^b g(s)(T_a^\alpha f)(s) d\alpha(s, a). \tag{17}$$

From Theorem 2.4 we have $I_a^\alpha (T_a^\alpha fg)(b) = f(b)g(b) - f(a)g(a)$, so the result follows.

□

Note that equation (17) of the previous theorem is also satisfied when f and g satisfy the hypothesis of Theorem 2.5.

2.3 Conformable variational calculus

Lazo and Torres [13] introduced the functional $\mathcal{J}$ as the variational integral

$$\mathcal{J}(y) = \int_a^b L\left(s, y(s), T_a^\alpha y(s)\right) d\alpha(s, a), \tag{18}$$

defined on the set of continuous functions $y : [a, b] \to \mathbb{R}$ such that $T_a^\alpha y$ exists in $[a, b]$, where the function $L : [a, b] \times \mathbb{R}^2 \to \mathbb{R}$ is of class C^1 in each of its arguments. The fundamental problem of the calculus of variations consists of finding the functions y for which the functional $\mathcal{J}$ reaches its extremum. Related approaches to conformable variational principles can also be found in [9, 19], where conformable derivatives are employed to extend classical variational and dynamical models.

For the purposes of this work, we rely on one key result from [13], which is stated as follows.

Lemma 2.1 (Fundamental lemma of conformable variational calculus)**.** *Let M be a continuous function on $[a, b]$. If*

$$\int_a^b \eta(s)M(s)\, d\alpha(s, a) = 0, \tag{19}$$

for any $\eta \in C[a, b]$ with $\eta(a) = \eta(b) = 0$, then $M(s) = 0$ for all $s \in [a, b]$.

Proof. Proceeding by contradiction, suppose that there exists $s_0 \in [a, b]$ such that $M(s_0) \neq 0$; without loss of generality, suppose that $M(s_0) > 0$. Since M is continuous, there exist $s_1, s_2 \in [a, b]$, such that $s_1 < s_2$, $s_0 \in [s_1, s_2]$ and $M(s) > 0$ for all $s \in [s_1, s_2]$. Let $\eta : [a, b] \to \mathbb{R}$ be defined by

$$\eta(s) = \begin{cases} (s - s_1)(s_2 - s), & \text{if } s \in [s_1, s_2], \\ 0, & \text{if } s \notin [s_1, s_2]. \end{cases} \tag{20}$$

This function is continuous on $[a, b]$ and satisfies that $\eta(a) = \eta(b) = 0$, furthermore

$$\int_a^b \eta(s)M(s)\, d\alpha(s, a) = \int_{s_1}^{s_2} (s - s_1)(s_2 - s)(s - a)^{\alpha-1} M(s)ds > 0, \tag{21}$$

since the integrand is positive on $[s_1, s_2]$, except at the endpoints; which contradicts the hypothesis. Therefore, $M(s) = 0$ for all $s \in [a, b]$.

□

The functions η in the previous lemma are called test functions. The conclusion of the lemma also holds if M is a piecewise continuous function and equation (19) is satisfied for test functions that are piecewise continuous and vanish at a and b. The proof is similar, since it is enough to choose test functions whose support is contained in the intervals of continuity of the function M. In fact, observing the function η used in the proof of the lemma, it is possible to restrict the test functions to those that are piecewise differentiable on $[a, b]$ with $\eta(a) = \eta(b) = 0$. One can even consider only functions of class C^1 that vanish at the endpoints of the interval.

Corollary 2.4. *Let $M \in C[a, b]$ and N be a differentiable function on $[a, b]$. If*

$$\int_a^b (\eta(s)M(s) + N(s)T_a^\alpha \eta(s))\, d\alpha(s, a) = 0, \tag{22}$$

for any $\eta \in C^1[a, b]$ such that $\eta(a) = \eta(b) = 0$, then $M(s) = T_a^\alpha N(s)$ for all $s \in [a, b]$.

Proof. Let M, N, and η be as in the hypothesis. By the linearity of the conformable integral and Theorem 2.6, we have

$$\begin{aligned} 0 &= \int_a^b \eta(s)M(s)d\alpha(s, a) + \int_a^b N(s)T_a^\alpha \eta(s)d\alpha(s, a) \\ &= \int_a^b \eta(s)M(s)d\alpha(s, a) + N(b)\eta(b) - N(a)\eta(a) - \int_a^b \eta(s)T_a^\alpha N(s)d\alpha(s, a) \\ &= \int_a^b \eta(s)\left[M(s) - T_a^\alpha N(s)\right] d\alpha(s, a). \end{aligned} \tag{23}$$

From the observation made after Lemma 2.1, it follows that $M(s) = T_a^\alpha N(s)$ for all $s \in [a, b]$.

□

By inspecting the arguments in the proof given above, it is possible to obtain a version of the previous corollary in which M is a piecewise continuous function on $[a, b]$, N is piecewise differentiable and it is assumed that equation (22) holds for every function η that is piecewise differentiable on $[a, b]$ such that $\eta(a) = \eta(b) = 0$. In that case, we conclude that $M(s) = T_a^\alpha N(s)$, except at a finite number of points in $[a, b]$.

3 Finite element method applied to a conformable differential equation

In this section, our goal is to obtain an approximate solution of the conformable differential equation

$$-T_0^\alpha(T_0^\alpha x)(t) + kx(t) = f(t), \tag{24}$$

where $f \in C[0, 1]$, $k > 0$, $\alpha \in (0, 1]$ and $x : [0, 1] \to \mathbb{R}$ is a function which satisfies the boundary condition $x(0) = x(1) = 0$.

When $\alpha = 1$, $f(t) = 0$ and $k < 0$ the equation (24) is known as the Helmholtz equation. When $\alpha = 1$, $f(t) = 0$ and $k = 0$, it is called the Poisson equation. For $\alpha \in (0, 1)$, if $f(t) = 0$ we shall call the equation (24) the conformable Helmholtz equation when $k = 1$, and the conformable Poisson equation when $k = 0$.

3.1 Finite element formulation

In [21] the finite element method is proposed using the test functions

$$\phi_i^\alpha(t) = \begin{cases} \frac{t^\alpha - t_{i-1}^\alpha}{t_i^\alpha - t_{i-1}^\alpha}, & t_{i-1} \le t \le t_i, \\ \frac{t_{i+1}^\alpha - t^\alpha}{t_{i+1}^\alpha - t_i^\alpha}, & t_i \le t \le t_{i+1}, \\ 0, & \text{otherwise.} \end{cases} \tag{25}$$

The approximation in that work is done in the Hilbert space framework by doing orthogonal projection on the finite dimensional subspaces generated by the functions ϕ_i^α. Similarly, in [7], the finite element method approximates a solution of a partial differential equation using test functions as cubic B-spline polynomials. In contrast, in this work, we shall use a simpler family of test functions, the hat functions.

We shall assume that there exists a solution of the equation (24) which satisfies that $T_0^\alpha x$ is differentiable. This happens automatically when x is twice differentiable.

Multiplying equation (24) by a piecewise differentiable test function on [0,1], η, with $\eta(0) = \eta(1) = 0$; calculating the left integral of order α and using the integration by parts formula, we get the following.

$$-\int_0^1 [T_0^\alpha(T_0^\alpha x)(t)\eta(t)]\, d\alpha(t, 0) + k\int_0^1 [x(t)\eta(t)]\, d\alpha(t, 0) = \int_0^1 f(t)\eta(t)d\alpha(t, 0), \tag{26}$$

$$\int_0^1 (T_0^\alpha x)(t)(T_0^\alpha \eta)(t)d\alpha(t, 0) + k\int_0^1 x(t)\eta(t)d\alpha(t, 0) = \int_0^1 f(t)\eta(t)d\alpha(t, 0). \tag{27}$$

That is, if x is a solution of the conformable differential equation (24) with $T_0^\alpha x$ differentiable, then it is also a solution of the conformable integral equation (27). We can say that a function x

is a weak solution of (24) if x is a solution of (27). In the rest of this section, we will be looking for a weak solution.

In order to get an approximate solution for the conformable integral equation above, we consider a regular partition of the interval $[0,1]$, $\{t_0, t_1, \ldots, t_N\}$, and we define for each $i \in \{1, \ldots, N-1\}$ the tent function (also known as a hat function) $h_i^N : [0,1] \to \mathbb{R}$ given by

$$h_i^N(t) = \begin{cases} N(t - t_{i-1}), & t \in [t_{i-1}, t_i], \\ N(t_{i+1} - t), & t \in [t_i, t_{i+1}], \\ 0, & \text{otherwise.} \end{cases} \tag{28}$$

It is well known that the family of functions

$$\left\{ \sum_{i=1}^{N-1} c_i h_i^N(t) : N \in \mathbb{N}, c_i \in \mathbb{R} \right\} \tag{29}$$

is a dense set on $C[0,1]$. In the following, we fix $N \in \mathbb{N}$, so we shall denote h_i^N simply by h_i.

We propose an approximate solution as

$$x_N(t) = \sum_{i=1}^{N-1} c_i h_i(t), \tag{30}$$

where $c_1, \ldots, c_{N-1}$ are parameters to be determined. Replacing this approximation in equation (27) we get

$$\sum_{i=1}^{N-1} c_i \int_0^1 (T_0^\alpha h_i)(t)(T_0^\alpha \eta)(t) d\alpha(t,0) + k \sum_{i=1}^{N-1} c_i \int_0^1 h_i(t)\eta(t) d\alpha(t,0) = \int_0^1 f(t)\eta(t) d\alpha(t,0). \tag{31}$$

In particular, this equation should be satisfied by the functions h_i. So, for $j \in \{1, \ldots, N-1\}$,

$$\sum_{i=1}^{N-1} c_i \left[\int_0^1 (T_0^\alpha h_i)(t)(T_0^\alpha h_j)(t) d\alpha(t,0) + k \int_0^1 h_i(t) h_j(t) d\alpha(t,0) \right] = \int_0^1 f(t) h_j(t) d\alpha(t,0). \tag{32}$$

Then we need to solve the above equation system of $N-1$ equations and $N-1$ unknowns.

We note that if $|i-j| \geq 2$ the functions h_i and h_j have disjoint support, thus we have the following cases:

- If $|i-j| \geq 2$, then the product $h_i(t)h_j(t)$ is identically zero.
- If $i = j$, then

$$h_i(t)h_j(t) = h_i^2(t) = \begin{cases} N^2(t - t_{i-1})^2, & t \in [t_{i-1}, t_i], \\ N^2(t_{i+1} - t)^2, & t \in [t_i, t_{i+1}], \\ 0, & \text{otherwise.} \end{cases} \tag{33}$$

- If $j = i+1$, then

$$h_i(t)h_{i+1}(t) = \begin{cases} N^2(t_{i+1} - t)(t - t_i), & t \in [t_i, t_{i+1}], \\ 0, & \text{otherwise.} \end{cases} \tag{34}$$

Furthermore,

$$(T_0^\alpha h_i)(t) = \begin{cases} Nt^{1-\alpha}, & t \in (t_{i-1}, t_i), \\ -Nt^{1-\alpha}, & t \in (t_i, t_{i+1}), \\ 0, & \text{otherwise.} \end{cases} \tag{35}$$

Having this in mind, if $|i-j| \geq 2$ then $(T_0^\alpha h_i)(T_0^\alpha h_j)$ is identically zero. When $i = j$, we have

$$(T_0^\alpha h_i)(t)(T_0^\alpha h_j)(t) = (T_0^\alpha h_i)^2(t) = \begin{cases} N^2 t^{2-2\alpha}, & t \in (t_{i-1}, t_i) \cup (t_i, t_{i+1}), \\ 0, & \text{otherwise.} \end{cases} \tag{36}$$

For $j = i+1$, we have

$$(T_0^\alpha h_i)(t)(T_0^\alpha h_{i+1})(t) = \begin{cases} -N^2 t^{2-2\alpha}, & t \in (t_i, t_{i+1}), \\ 0, & \text{otherwise.} \end{cases} \tag{37}$$

Consequently,

$$\int_0^1 (T_0^\alpha h_i)(t)(T_0^\alpha h_j)(t) d\alpha(t,0) = \int_0^1 h_i(t)h_j(t) d\alpha(t,0) = 0, \tag{38}$$

when $|i-j| \geq 2$. For $i = j$, we get

$$\begin{aligned} \int_0^1 (T_0^\alpha h_i)(t)(T_0^\alpha h_j)(t) d\alpha(t,0) &= N^2 \int_{t_{i-1}}^{t_{i+1}} t^{2-2\alpha} t^{\alpha-1} dt = N^2 \int_{t_{i-1}}^{t_{i+1}} t^{1-\alpha} dt \\ &= \frac{N^2}{2-\alpha} \left[\left(\frac{i+1}{N} \right)^{2-\alpha} - \left(\frac{i-1}{N} \right)^{2-\alpha} \right] \\ &= \frac{N^\alpha}{2-\alpha} \left[(i+1)^{2-\alpha} - (i-1)^{2-\alpha} \right]. \end{aligned} \tag{39}$$

To compute the diagonal term we use that the support of h_i is the interval $[t_{i-1}, t_{i+1}]$. Thus,

$$\int_0^1 h_i(t)h_i(t)\, d\alpha(t,0) = N^2 \left[\int_{t_{i-1}}^{t_i} (t - t_{i-1})^2 t^{\alpha-1}\, dt + \int_{t_i}^{t_{i+1}} (t_{i+1} - t)^2 t^{\alpha-1}\, dt \right]. \tag{40}$$

Expanding the quadratic terms and integrating term by term yields

$$\begin{aligned} \int_0^1 h_i(t)h_i(t)\, d\alpha(t,0) = N^2 \Bigg\{ & \frac{t_{i+1}^{\alpha+2} - t_{i-1}^{\alpha+2}}{\alpha+2} - \frac{2}{\alpha+1} \big[t_{i-1}(t_i^{\alpha+1} - t_{i-1}^{\alpha+1}) + t_{i+1}(t_{i+1}^{\alpha+1} - t_i^{\alpha+1}) \big] \\ & + \frac{1}{\alpha} \big[t_{i-1}^2 (t_i^\alpha - t_{i-1}^\alpha) + t_{i+1}^2 (t_{i+1}^\alpha - t_i^\alpha) \big] \Bigg\}. \end{aligned} \tag{41}$$

Using $t_k = \frac{k}{N}$ and simplifying, we obtain

$$\int_0^1 h_i(t)h_i(t)\, d\alpha(t,0) = \frac{1}{\alpha(\alpha+1)(\alpha+2)N^\alpha} \left\{ 2\left[(i+1)^{\alpha+2} - (i-1)^{\alpha+2} \right] - 4(\alpha+2) i^{\alpha+1} \right\}. \tag{42}$$

For $j = i+1$, we get

$$\begin{aligned} \int_0^1 (T_0^\alpha h_i)(t)(T_0^\alpha h_j)(t) d\alpha(t,0) &= -N^2 \int_{t_i}^{t_{i+1}} t^{2-2\alpha} t^{\alpha-1} dt = -N^2 \int_{t_i}^{t_{i+1}} t^{1-\alpha} dt \\ &= -\frac{N^2}{2-\alpha} \left[\left(\frac{i+1}{N} \right)^{2-\alpha} - \left(\frac{i}{N} \right)^{2-\alpha} \right] \\ &= -\frac{N^\alpha}{2-\alpha} \left[(i+1)^{2-\alpha} - i^{2-\alpha} \right], \end{aligned} \tag{43}$$

and

$$
\begin{aligned}
\int_0^1 h_i(t)h_j(t)d\alpha(t,0) &= N^2\int_{t_i}^{t_{i+1}}(t_{i+1}-t)(t-t_i)t^{\alpha-1}dt \\
&= N^2\int_{t_i}^{t_{i+1}}[-t^2+(t_i+t_{i+1})t-t_it_{i+1}]t^{\alpha-1}dt \\
&= N^2\left[-\frac{t^{\alpha+2}}{\alpha+2}+\frac{(t_i+t_{i+1})t^{\alpha+1}}{\alpha+1}-\frac{t_it_{i+1}t^{\alpha}}{\alpha}\right]_{t_i}^{t_{i+1}} \\
&= \frac{1}{N^\alpha}\left[\frac{i^{\alpha+2}-(i+1)^{\alpha+2}}{\alpha+2}+\frac{2i+1}{\alpha+1}\left[(i+1)^{\alpha+1}-i^{\alpha+1}\right]\right. \\
&\qquad \left.-\frac{i(i+1)}{\alpha}\left[(i+1)^{\alpha}-i^{\alpha}\right]\right]. \qquad (44)
\end{aligned}
$$

Finally, we have

$$
\int_0^1 f(t)h_j(t)d\alpha(t,0) = N\left[\int_{t_{i-1}}^{t_i} f(t)(t-t_{i-1})t^{\alpha-1}dt+\int_{t_i}^{t_{i+1}} f(t)(t_{i+1}-t)t^{\alpha-1}dt\right]. \qquad (45)
$$

From the above, the matrix associated with the linear system (32) is tridiagonal.

The ideas put forth in this section can be applied on a general interval $[a,b]$ and with boundary conditions different from 0, by considering two more hat functions

$$
h_0(t)=\begin{cases} N(t_1-t), & t\in[t_0,t_1], \\ 0, & \text{otherwhise,}\end{cases} \qquad (46)
$$

and

$$
h_N(t)=\begin{cases} N(t-t_{N-1}), & t\in[t_{N-1},t_N], \\ 0, & \text{otherwhise.}\end{cases} \qquad (47)
$$

3.2 Numerical examples

In this section we illustrate the numerical behavior of the finite element formulation through a series of computational experiments. We consider the boundary value problem involving the conformable derivative

$$
-T_0^\alpha(T_0^\alpha x)(t)+kx(t)=f(t), \qquad (48)
$$

where $f\in C[0,1]$, $k>0$, $\alpha\in(0,1]$ and $x:[0,1]\to\mathbb{R}$ satisfies the boundary conditions $x(0)=x(1)=0$.

The approximations are computed for the derivative orders $\alpha=0.2,0.4,0.6,0.8$, and 1, using different discretizations of the interval $[0,1]$. To visualize the numerical solutions we consider partitions with $N=5,10,20$, and 40 subintervals, which allows the influence of mesh refinement on the computed approximations to be observed. The numerical experiments were carried out in MATLAB R2025b, which was used to compute the finite element approximations and generate the corresponding figures.

By Theorem 2.3, the conformable equation can be rewritten as a second–order ordinary differential equation whose coefficients depend on powers of t. For $\alpha\in(0,1)$ the resulting equation has variable coefficients with singular behavior at $t=0$. In general, equations of this type do not admit explicit expressions in terms of elementary functions, so a direct analytical comparison is typically unavailable. In this setting, numerical approximations provide useful information about the behavior of the solutions.

To complement the graphical results, we also examine the behavior of the method under mesh refinement through a numerical convergence study. For this purpose, the discretizations $N = 5, 10, 20, 40, 80$, and 160 are considered. Let $x_N(t)$ denote the finite element approximation obtained with N subintervals. The error is measured in the maximum norm

$$E(N) = \max_{t\in[0,1]} |x_N(t) - x_{\text{ref}}(t)|\,, \tag{49}$$

where $x_{\text{ref}}(t)$ is taken as a reference solution.

The errors are evaluated on a uniform grid of the interval $[0, 1]$. Specifically, we consider the set

$$T = \{t_i = \tfrac{i-1}{1999} : i = 1, \ldots, 2000\},$$

and approximate the error by

$$E(N) = \max_{t_i\in T} |x_N(t_i) - x_{\text{ref}}(t_i)|\,. \tag{50}$$

3.2.1 Example 1

As a first numerical test we consider the boundary value problem

$$-T_0^\alpha(T_0^\alpha x)(t) + x(t) = t^{\alpha+1} - t^\alpha - (\alpha + 1)t^{1-\alpha}, \tag{51}$$

where $\alpha \in (0, 1]$ and $x : [0, 1] \to \mathbb{R}$ satisfies the boundary conditions $x(0) = x(1) = 0$.

For this problem the exact solution is known. Indeed, direct computation shows that

$$x_{\text{ref}}(t) = t^\alpha(t - 1) \tag{52}$$

satisfies equation (51). This makes it possible to evaluate the accuracy of the numerical scheme by comparing the finite element approximations with the analytical solution.

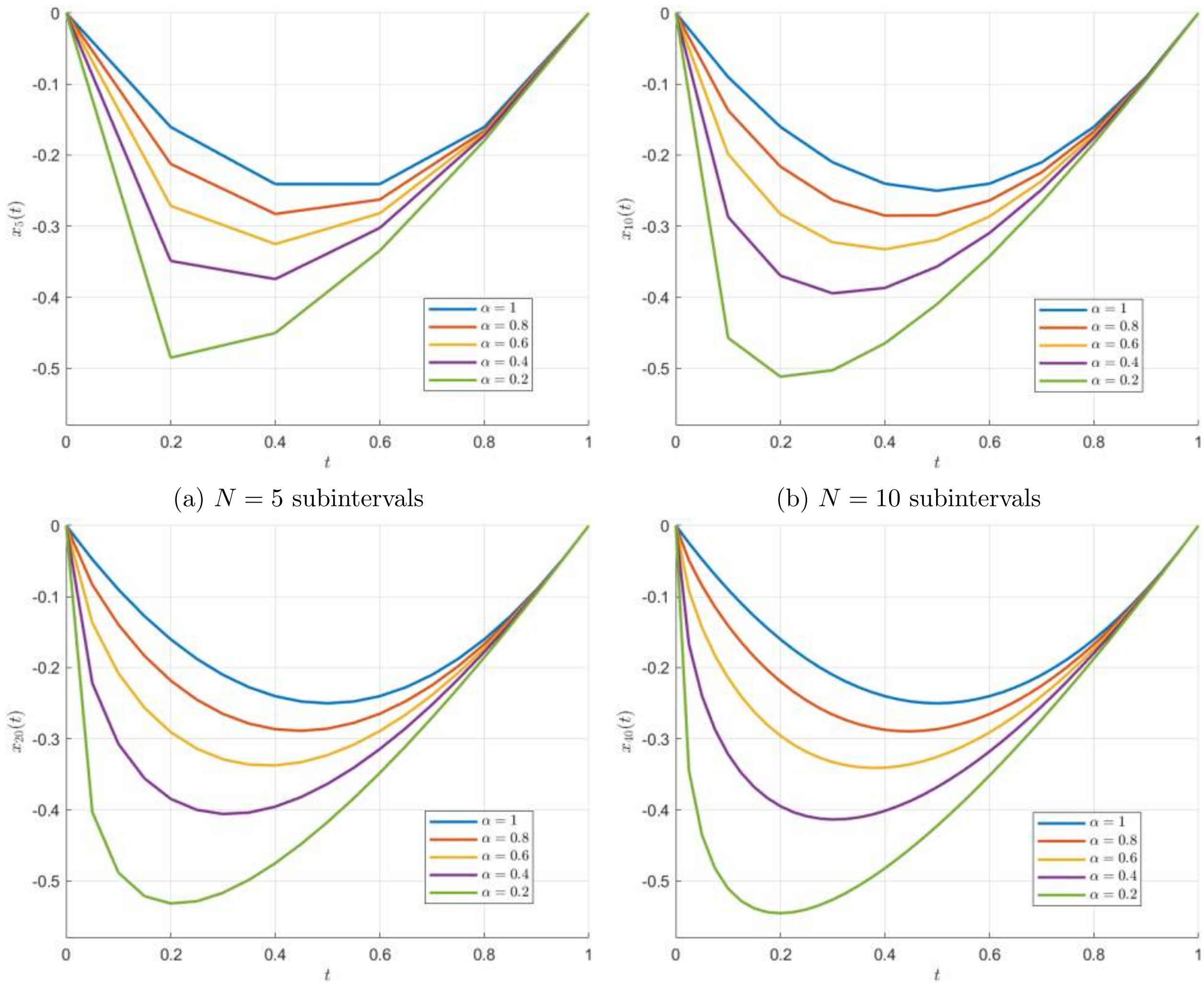
Figure 2 displays the approximations $x_N(t)$ obtained for several values of α and different mesh discretizations. When $\alpha = 1$ the conformable model reduces to the classical equation

$$-x''(t) + x(t) = t^2 - t - 2. \tag{53}$$

The numerical errors are computed with respect to the analytical solution $x_{\text{ref}}(t)$. Table 1 reports the maximum errors for the discretizations $N = 5, 10, 20, 40, 80$ and 160.

Table 1: Maximum errors $E(N)$ for different mesh refinements and derivative orders α for equation (51).

N	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1$
5	0.40879	0.200430	0.093761	0.0370490	0.00975740
10	0.35747	0.152540	0.059815	0.0182980	0.00246570
20	0.31377	0.117290	0.039070	0.0096384	0.00062044
40	0.27595	0.090499	0.025786	0.0052842	0.00015566
80	0.24294	0.069842	0.017077	0.0029635	3.8930e-05
160	0.21379	0.053792	0.011316	0.0016820	9.7414e-06



(a) $N = 5$ subintervals

(b) $N = 10$ subintervals

(c) $N = 20$ subintervals

(d) $N = 40$ subintervals

Figure 2: Finite element approximations $x_N(t)$ of the solution of (51) with boundary conditions $x(0) = x(1) = 0$, for different values of the order $\alpha = 0.2, 0.4, 0.6, 0.8, 1$ and several mesh discretizations.

3.2.2 Example 2

We next consider a second problem in order to further examine the performance of the method. In this case we study the equation

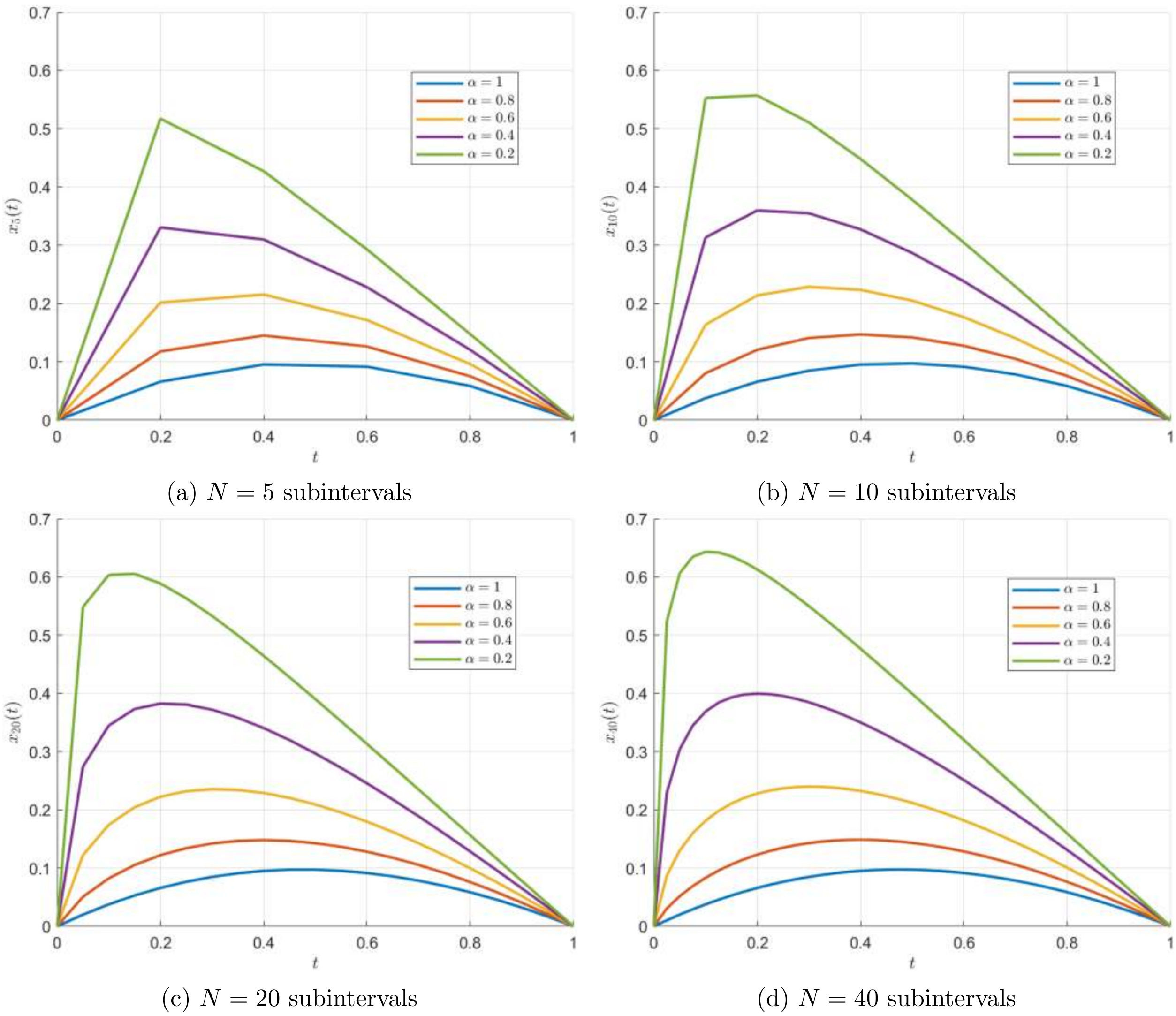
$$-T_0^\alpha(T_0^\alpha x)(t) + x(t) = \cos(t), \tag{54}$$

where $\alpha \in (0, 1]$ and $x : [0, 1] \to \mathbb{R}$ satisfies the boundary conditions $x(0) = x(1) = 0$.

Figure 3 presents the finite element approximations computed for several values of α and different mesh discretizations. As before, when $\alpha = 1$ the equation coincides with the classical problem

$$-x''(t) + x(t) = \cos(t). \tag{55}$$

In this example an analytical solution is not available. For the computation of the errors we therefore take as reference the approximation obtained with the finest discretization, namely $N = 160$.



(a) $N = 5$ subintervals

(b) $N = 10$ subintervals

(c) $N = 20$ subintervals

(d) $N = 40$ subintervals

Figure 3: Finite element approximations $x_N(t)$ of the solution of (54) with boundary conditions $x(0) = x(1) = 0$, for $\alpha = 0.2, 0.4, 0.6, 0.8, 1$ and several mesh discretizations.

Table 2 lists the values of the maximum error $E(N)$ for different mesh refinements.

3.2.3 Discussion

The two examples illustrate similar numerical behavior. In both cases the finite element approximations become progressively smoother as the mesh is refined, and the numerical solutions tend to stabilize as the number of subintervals increases. The error tables confirm this observation, showing a consistent decrease of the error when the discretization is refined.

Another common feature is the dependence of the solution on the order α. As α approaches 1, the conformable model approaches the corresponding classical differential equation. The computed solutions reflect this transition and display profiles similar to those of the classical problem.

The first example allows a direct comparison with an analytical solution, which confirms the accuracy of the scheme. The second example shows that the same behavior persists even when an exact solution is not available and the reference solution is obtained numerically. Taken together,

Table 2: Maximum errors $E(N)$ for different mesh refinements and derivative orders α for equation (54).

N	$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$	$\alpha = 1$
5	0.57457	0.261660	0.092214	0.0257430	0.00468520
10	0.50100	0.200340	0.058807	0.0126440	0.00120880
20	0.41706	0.144900	0.036111	0.0063214	0.00030722
40	0.31970	0.094474	0.020733	0.0031294	7.7455e-05
80	0.20104	0.054569	0.010684	0.0014729	1.8687e-05

these results indicate that the proposed finite element formulation provides stable and consistent approximations for different values of the conformable order.

4 Conclusions

In this work we proposed a finite element formulation for conformable differential equations. The approach is based on a weak formulation obtained from a version of the fundamental lemma of the calculus of variations adapted to the conformable derivative. The discretization of the domain, together with the use of piecewise-defined test functions, allows the method to be implemented within the standard finite element framework. In this way, conformable operators can be incorporated into a variational setting while preserving the usual numerical structure of the method.

The numerical experiments illustrate the applicability of the proposed formulation for different values of the order of derivation. The computed approximations improve as the mesh is refined, which is consistent with the convergence behavior observed in the error study. In the example where an analytical solution is available, the numerical results show good agreement with the exact solution. The results also indicate a gradual variation of the numerical solutions as the parameter α changes. In particular, when $\alpha = 1$ the conformable model reduces to the classical differential equation, and the finite element approximation reproduces the qualitative behavior of the corresponding classical solution.

The local structure of the conformable derivative facilitates its incorporation into standard numerical schemes and makes the proposed formulation suitable for several extensions. Possible directions include problems posed on more general domains, equations with more complex boundary conditions, and models involving piecewise differentiable solutions. From an analytical perspective, it would be of interest to identify conditions guaranteeing existence and uniqueness of weak solutions and to determine whether a result in the spirit of the Lax–Milgram theorem can be established in the conformable setting. Another related question concerns situations in which the weak formulation leads back to the original differential equation beyond the case where $T_0^\alpha x$ is differentiable. From the numerical side, further work includes the study of the approximation properties of the finite element scheme in the associated test-function spaces, as well as the consideration of more demanding numerical examples, such as problems involving piecewise differentiable solutions.

5 Conflict of interest

The authors declare no conflicts of interest.

6 Artificial Intelligence Usage Statement

The authors declare that they have not used any generative artificial intelligence applications, software or websites for the writing of the present paper, the design of tables and figures or the analysis and interpretation of the data.

7 Code Availability

The MATLAB code used in the numerical experiments is available from the corresponding author upon reasonable request.

8 Graphical Abstract

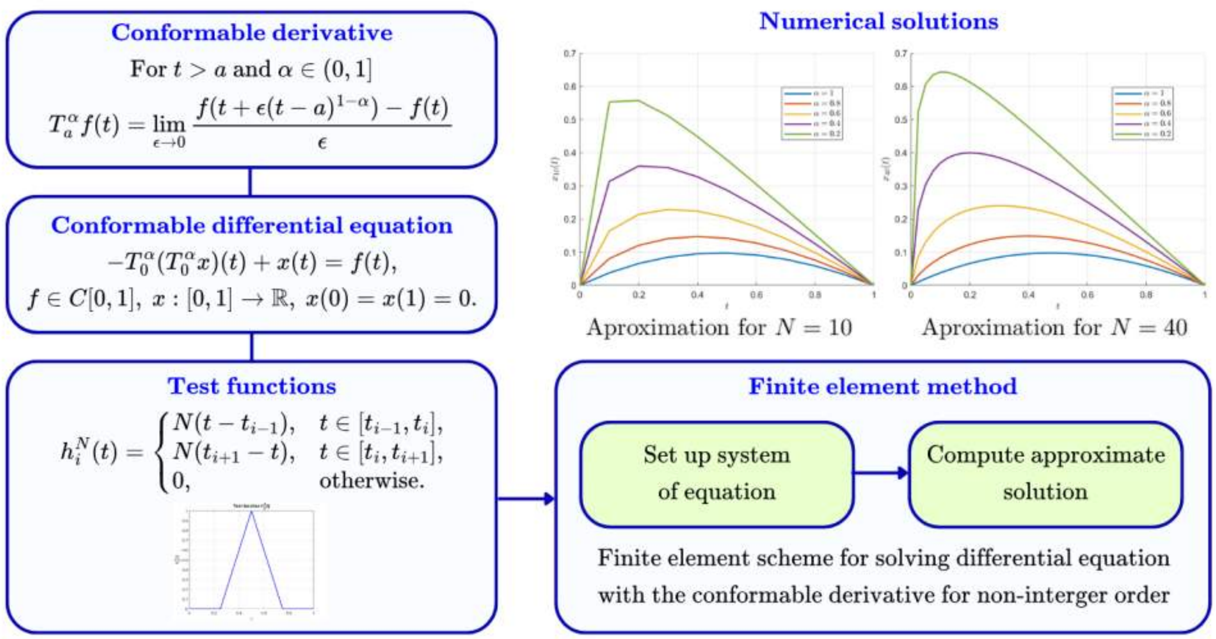


Figure 4: Graphical abstract.

9 Contribution Roles

ROLE	AUTHORS
Conceptualization	P. Toledo C. A. Hernández-Linares M. L. Avendaño-Garrido
Formal analysis	P. Toledo C. A. Hernández-Linares M. L. Avendaño-Garrido
Investigation	P. Toledo C. A. Hernández-Linares M. L. Avendaño-Garrido
Methodology	P. Toledo C. A. Hernández-Linares M. L. Avendaño-Garrido
Software	P. Toledo C. A. Hernández-Linares M. L. Avendaño-Garrido
Validation	P. Toledo C. A. Hernández-Linares M. L. Avendaño-Garrido
Visualization	P. Toledo C. A. Hernández-Linares M. L. Avendaño-Garrido
Writing - original draft	P. Toledo C. A. Hernández-Linares M. L. Avendaño-Garrido
Writing - review & editing	P. Toledo C. A. Hernández-Linares M. L. Avendaño-Garrido

References

[1] T. Abdeljawad. On conformable fractional calculus. *Journal of Computational and Applied Mathematics*, 279:57–66, 5 2015.

[2] A. A. Alderremy, H. I. Abdel-Gawad, K. M. Saad, and E. M. Aly Shaban. New exact solutions of time conformable fractional Klein Kramer equation. *Optical and Quantum Electronics*, 53(12):693, dec 2021.

[3] A. Arafa. A different approach for conformable fractional biochemical reaction—diffusion models. *Applied Mathematics-A Journal of Chinese Universities*, 35(4):452–467, oct 2020.

[4] J.A. Burns. *Introduction to the Calculus of Variations and Control with Modern Applications*. Chapman & Hall/CRC Applied Mathematics and Nonlinear Science. CRC Press, 2013.

[5] J. M. Cruz-Duarte, J. J. Rosales-García, and C. R. Correa-Cely. Entropy Generation in a Mass-Spring-Damper System Using a Conformable Model. *Symmetry*, 12(3):395, mar 2020.

[6] K. Diethelm. *The Analysis of Fractional Differential Equations*, volume 2004 of *Lecture Notes in Mathematics*. Springer Berlin Heidelberg, Berlin, Heidelberg, 2010.

[7] A. R. Hadhoud, F. E. I. Abd Alaal, and T. Radwan. Modified finite element numerical method for solving conformable space-time fractional nonlinear partial differential equations. *Fractals*, 30(10):2240247, 2022.

[8] I. Haouam. On the conformable fractional derivative and its applications in physics. *Journal of Theoretical and Applied Physics*, 18(6), 2024.

[9] B. B. Iskender and D. Yapışkan. Generalized conformable variational calculus and optimal control problems with variable terminal conditions. *AIMS Mathematics*, 5(2):1105–1126, 2020.

[10] R. Kaviya and P. Muthukumar. Dynamical analysis and optimal harvesting of conformable fractional prey–predator system with predator immigration. *The European Physical Journal Plus*, 136(5):542, may 2021.

[11] R. Khalil, M. Al Horani, R. Yousef, and M. Sababheh. A new definition of fractional derivative. *Journal of Computational and Applied Mathematics*, 264:65–70, 7 2014.

[12] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo. *Theory and Applications of Fractional Differential Equations*. Elsevier B.V., Amsterdam, 2006.

[13] M. J. Lazo and D. F.M. Torres. Variational calculus with conformable fractional derivatives. *IEEE/CAA Journal of Automatica Sinica*, 4:340–352, 4 2017.

[14] C. Li and F. Zeng. *Numerical Methods for Fractional Calculus*. Chapman & Hall/CRC Numerical Analysis and Scientific Computing. CRC Press, 2015.

[15] W. K. Liu, S. Li, and H. S. Park. Eighty Years of the Finite Element Method: Birth, Evolution, and Future. *Archives of Computational Methods in Engineering*, 29(6):4431–4453, oct 2022.

[16] L. Martínez, J.J. Rosales, C.A. Carreño, and J.M. Lozano. Electrical circuits described by fractional conformable derivative. *International Journal of Circuit Theory and Applications*, 46(5):1091–1100, may 2018.

[17] A. Martynyuk, G. Stamov, I. Stamova, and E. Gospodinova. Formulation of Impulsive Ecological Systems Using the Conformable Calculus Approach: Qualitative Analysis. *Mathematics*, 11(10):2221, may 2023.

[18] K. S. Miller and B. Ross. *An Introduction to the Fractional Calculus and Fractional Differential Equations*. John Wiley & Sons Inc., New York, 1993.

[19] S. Ogrecki and S. Asliyüce. Fractional variational problems on conformable calculus. *Communications Faculty Of Science University of Ankara Series A1Mathematics and Statistics*, 70(2):719–730, dec 2021.

[20] J. Rosales-García, J. A. Andrade-Lucio, and O. Shulika. Conformable derivative applied to experimental Newton's law of cooling. *Revista Mexicana de Física*, 66(2 Mar-Apr):224–227, mar 2020.

[21] L. Sadek, T. A. Lazăr, and I. Hashim. Conformable finite element method for conformable fractional partial differential equations. *AIMS Mathematics*, 8(12):28858–28877, 2023.

[22] G. Sales Teodoro, J. A. Tenreiro Machado, and E. Capelas de Oliveira. A review of definitions of fractional derivatives and other operators. *Journal of Computational Physics*, 388:195–208, 7 2019.

[23] K. Soni and A. K. Sinha. Dynamics of epidemic model with conformable fractional derivative. *Nonlinear Science*, 4:100040, sep 2025.

[24] Ş. Toprakseven. Numerical Solutions of Conformable Fractional Differential Equations by Taylor and Finite Difference Methods. *Süleyman Demirel Üniversitesi Fen Bilimleri Enstitüsü Dergisi*, 23(3):850–863, dec 2019.

[25] S. A. Zanib, T. Zubair, S. Ramzan, M. B. Riaz, M. I. Asjad, and M. Taseer. A conformable fractional finite difference method for modified mathematical modeling of SAR-CoV-2 (COVID-19) disease. *PLOS ONE*, 19(10):e0307707, oct 2024.