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Diseño educativo para la enseñanza de la cinemática de mecanismo

-Perera-Cortez et al.

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En el número 34 del Journal of Basic Sciences se presentan ocho contribuciones que reflejan el carácter multidisciplinario de la revista, al abordar temas relacionados con ingeniería y ciencias de materiales, enseñanza de las ciencias, farmacéutica, física y matemáticas. Desde sus respectivas áreas, los trabajos emplean herramientas científicas para ampliar el conocimiento, fortalecer los procesos de enseñanza-aprendizaje y contribuir a la solución de problemáticas de interés social, tecnológico y académico.

El número inicia con el artículo de López-Zapata et al., en el que se evalúa la corrosión atmosférica del acero al carbono y cobre en el municipio de Cunduacán, Tabasco, considerando la presencia de contaminantes atmosféricos asociados al entorno. Los resultados destacan la importancia de fortalecer las estrategias de control de emisiones y protección de materiales empleados en infraestructura industrial y residencial, particularmente en regiones con actividad petrolera.

Posteriormente, se presentan dos contribuciones relacionadas con la enseñanza de las ciencias. Perera Cortez et al. desarrollan y validan un prototipo didáctico de un mecanismo biela-manivela con corredera para la enseñanza de la cinemática en estudiantes de ingeniería. El dispositivo integra teoría, simulación y mediciones experimentales, favoreciendo una comprensión más interactiva de los conceptos estudiados. Por su parte, López-de Dios et al. analizan el uso y valoración de herramientas informáticas por profesores de la Licenciatura en Química de la Universidad Juárez Autónoma de Tabasco durante la transición hacia la enseñanza virtual ocasionada por la pandemia de COVID-19. Sus resultados resaltan la necesidad de fortalecer no solamente las competencias técnicas digitales, sino también su adecuada integración pedagógica en la enseñanza de disciplinas experimentales.

En el área farmacéutica, Díaz-Martínez et al. comparan la ciclofosfamida y la bendamustina para el tratamiento del cáncer desde una perspectiva farmacoterapéutica y farmacoeconómica. El estudio considera eficacia, costos, reacciones adversas y disponibilidad de los medicamentos, proporcionando elementos para apoyar la toma de decisiones clínicas. Los resultados muestran ventajas económicas para la ciclofosfamida, mientras que la bendamustina representa una alternativa cuando se priorizan aspectos relacionados con tolerabilidad y seguridad.

Dentro del área de física, Rendón et al. estudian la variación de la presión atmosférica con respecto a la altura mediante modelos que incorporan un gradiente lineal de temperatura en las ecuaciones de estado de los gases ideales y de van der Waals. Esta aproximación permite representar de manera más realista el comportamiento de la atmósfera terrestre y explorar posibles comportamientos en Venus, Marte y Titán.

El número incluye además tres contribuciones del área de matemáticas. Balcázar-Araiza y Ojeda presentan una generalización dentro del Cálculo Fraccionario Multivariable mediante la definición y análisis de integrales y diferenciales de Riemann-Liouville de tipo (α, k, P), estableciendo propiedades y teoremas fundamentales asociados a estos operadores. Toledo et al. desarrollan un método numérico basado en elementos finitos para resolver ecuaciones diferenciales unidimensionales que involucran la derivada conformable, ofreciendo una herramienta para la aproximación eficiente de este tipo de problemas. Finalmente, Remigio-Juárez et al. presentan una introducción a los complejos de Vietoris-Rips y a la homología persistente, incluyendo el uso de diagramas de persistencia y herramientas computacionales en Python para el análisis de conjuntos de datos, con posibles aplicaciones en diversas áreas científicas.

De este modo, el número 34 del Journal of Basic Sciences reúne investigaciones y contribuciones académicas que muestran la diversidad de enfoques de las ciencias básicas y sus aplicaciones. En conjunto, los artículos fortalecen la difusión y divulgación del conocimiento científico y buscan generar impacto en los ámbitos regional, nacional e internacional.

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On Riemann-Liouville Operators of type (α, k, P) and Related Function Spaces
Sobre operadores Riemann-Liouville de tipo (α, k, P) y espacios de funciones asociados

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Abstract
In this paper, a first study of the **integrals and differentials of type** (α, k, P) was carried out. A generalization of the k-fractional operators proposed in previous works for one-variable functions was obtained, alongside with the trajectory fractional recently proposed. Relevant associated function spaces were proposed and studied, as well as basic properties concerning this generalization. Moreover, it was proven that several known fractional derivatives and integrals are particular cases of our proposal and finally, we complemented with fundamental theorems of calculus for these operators.

***Keywords**: Fractional Integral, Fractional Derivative, k-Riemann-Liouville operators, (α, k, P)-Riemann-Liouville operators, function spaces, path choices, arc-length functions.*

Resumen
En este trabajo se llevó a cabo una primera investigación sobre las **integrales y diferenciales de Riemann-Liouville de tipo** (α, k, P). Se obtuvo una generalización de los operadores k-fraccionarios para funciones de una variable planteados en trabajos previos, en conjunto los con operadores fraccionarios de trayectoria para funciones de varias variables recientemente propuestos. Se definieron los espacios de funciones relevantes asociados a dichos operadores y se exploraron primeras propiedades sobre los mismos. Asimismo, se proporcionaron ejemplos de derivadas e integrales fraccionarias conocidas que resultan como casos particulares y finalmente, se complementó el estudio con los teoremas fundamentales del cálculo correspondientes a estos operadores.

***Palabras claves**: Integral fraccional, derivada fraccional, k-operadores de tipo Riemann-Liouville, (α, k, P)-operadores de tipo Riemann-Liouville, espacios de funciones, elección de trayectorias, funciones de longitud de arco.*

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1 Introduction

Fractional Calculus consists of the analysis and application of differential and integral operators of arbitrary real or complex orders. What is more, due to the efforts of many other researchers on the generalization and abstraction of these ideas [1], [2], [3], we could affirm that Fractional Calculus intersects with the investigation of general linear operators defined on function spaces of interest, which generalize the concept of derivatives and integrals. Its origins trace back to the time of L'Hôpital and Leibniz and since then, the field has experienced a steady but constant growth (see [4] for a historical overview or [5, Ch. 1] for a complement). To the present, Fractional Calculus has proven to be useful to model a wide of phenomena, featuring population dynamics [6], [7], [8], electrical circuits [9], [10], [11], epidemics [12], [13], [14], classical mechanics [15], [16], [17], anomalous diffusion [18], [19], [20] among others. The memory and non-locality properties of certain fractional operators [21], along with their potential to enhance the precision of models that traditionally rely on ordinary derivatives [22], [23], have brought within an increase of the trend on recurring to the fractional approach.

On the theoretical front, while substantial work has been done to demonstrate the potential of fractional operators across various fields [4], [24], [25], much remains to be explored. For instance, a comprehensive and unified theory of Fractional Differential Geometry has yet to be developed and consequently, many improvements are still to be done in the realm of Multivariable Fractional Calculus. In this sense, many efforts have been carried out [4], [24], [26] to explore different avenues for the development of this area; in particular, Balcázar-Araiza et al [27] proposed a trajectory-dependent approach that generalizes both the one-variable Riemann-Liouville fractional operators and the integral of a scalar field w.r.t. the arc-length of a regular path, which showcased both theoretical and applied potential for further research and modeling (e.g. [28]). Simultaneously, in articles such as [29], [30] an extension of the one-variable Riemann-Liouville derivatives was implemented via the addition of an additional parameter and the incorporation of the generalized k-Gamma function [31]. Hence, our goal in this project is to unify both approaches to devise a more general object known as the Riemann-Liouville integral of type (α, k, P), which encompasses each of the aforementioned fractional operators in the realm of several-variables functions.

2 One-Variable Riemann-Liouville k-Fractional Calculus

Notation 2.1. *From now on, unless anything else is specified, we will denote* $\mathbb{R}_{>0} := (0,\infty)$ *and* $\mathbb{R}_{\geq 0} := [0,\infty)$.

In [31], Diaz and Pariaguan, while studying problems related to combinatorics and the perturbative computation of Feynman integrals, introduced the **generalized k-Gamma function** $\Gamma_k(x)$, a one-parameter generalization of the classical gamma function $\Gamma(x)$ with the property $\Gamma_1(x) = \Gamma(x)$. Shortly after, a variety of new extensions of the classical fractional operators were introduced in the literature, nowadays known as **k-fractional operators**.

Definition 2.1 ([31]). *Let $k \in \mathbb{R}_{>0}$ and $z \in \mathbb{C}$. The k-**Gamma function** $\Gamma_k(z)$ is defined as*

$$\Gamma_k(z) = \int_0^\infty e^{-\frac{t^k}{k}} t^{z-1} dt, \quad \mathrm{Re}(z) > 0.$$

Notice that when $k \to 1$, we have $\Gamma_k(z) \to \Gamma(z)$, where $\Gamma(\cdot)$ denotes the classical Gamma function. Moreover, we have the following properties whose proof is contained in [31].

Proposition 2.1. *Let $k \in \mathbb{R}_{>0}$. The k-Gamma function satisfies the following properties:*

(i) $\Gamma_k(z + k) = z\Gamma_k(z)$.

(ii) If $\Gamma(\cdot)$ is the classical Gamma function, then

$$\Gamma_k(z) = k^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right), \quad z \in \mathbb{C}, \mathrm{Re}(z) > 0. \tag{1}$$

In [29], Mubeen and Habbibulah introduced the *k-Riemann-Liouville fractional integral*, subsequently extended to the *k-Weyl-Liouville fractional integral* in [32].

Definition 2.2. *Let α, $k \in \mathbb{R}_{>0}$ and $f \in L([a,b])$, $-\infty \leq a < b \leq \infty$. The k-**Riemman-Liouville fractional integral of order** α is defined by*

$$I_{a+}^{(\alpha,k)} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt, \quad x \in [a,b]. \tag{2}$$

Observe that, due to relation (4) and [4, Theorem 2.7], the operator $I_{a+}^{(\alpha,k)}$ converges (in the pointwise sense) to I_{a+}^{α} as $k \to 1$, i.e., we recover the classical Riemann-Liouville fractional integral of order α ([5], [33]).

The following definition is adapted from [30, Definition 6].

Definition 2.3. *Let $\alpha, k \in \mathbb{R}_{>0}$ and $n \in \mathbb{N}$ such that $n = [\alpha/k] + 1$, $f \in L([a,b])$ and $I_k^{nk-\alpha} f(t) \in C^n([a,b])$; the k-**Riemann-Liouville fractional derivative of order** α is given by*

$$D_{a+}^{(\alpha,k)} f(x) := \left(\frac{d}{dt}\right)^n k^n I_{a+}^{(nk-\alpha,k)} f(x), \quad x \in [a,b]. \tag{3}$$

Moreover, according to [30, eq. (I.11)], we can express $I_{a+}^{(\alpha,k)}$ in terms of the classical Riemann-Liouville fractional integral as

$$I_{a+}^{(\alpha,k)} = k^{-\frac{\alpha}{k}} I_{a+}^{\alpha/k}. \tag{4}$$

From equation (3) and (4), it also follows that

$$D_{a+}^{(\alpha,k)} = k^{\frac{\alpha}{k}} D_{a+}^{\alpha/k}. \tag{5}$$

Here, I_{a+}^{ν} and D_{a+}^{ν} are the classical Riemann-Liouville fractional operators of order ν ([5], [33]).

3 Path Choices

Notation 3.1. *For the rest of the paper, given some intervals $[a,b], [c,d] \subseteq \mathbb{R}$, let us set*

$$\mathrm{Diff}_+([a,b],[c,d]) := \{f : [a,b] \to [c,d] : \; f, f^{-1} \in C^1([a,b],[c,d]) \text{ and } f' > 0\}.$$

Moreover, for $[a,b] = [c,d]$ we let $\mathrm{Diff}_+([a,b]) := \mathrm{Diff}_+([a,b],[a,b])$.

In [27, Definition 2.3], trajectory-dependent generalizations for the Riemann-Liouville fractional integrals were given; however, not only a positive order but also a trajectory must be given to evaluate the integral of the given scalar field along such curve. If we repeat this process for every point $\mathbf{x}$ in the domain into consideration, we are implicitly assigning a parametric curve to every $\mathbf{x}$ to be used to compute our fractional trajectory integral. We formalize this idea as follows: first, let us define the **set of C^j-class simple regular curves parametrized on $[0,1]$ ending on points of U as**

$$\mathfrak{C}^j(U) := \{\gamma \in C^j([0,1],\mathbb{R}^d) : \gamma' \neq \mathbf{0}, \; \gamma \text{ is one-to-one and } \gamma(1) = \mathbf{x}, \text{ for some } \mathbf{x} \in U\}.$$

Definition 3.1. *A j-**th order path choice** on U is a function $P : U \to \mathfrak{C}^j(U)$, such that*

(PC) for every $\mathbf{x} \in U$ *and every* $t \in [0,1]$ *we have* $Im(P(P(\mathbf{x})(t))) \subseteq Im(P(\mathbf{x}))$.

For $j = \infty$, $P : U \to \mathfrak{C}^{\infty}(U)$ *is a* ***smooth order path choice on*** U***.*** *Moreover, we use the notation* $P(\mathbf{x}) = \gamma_{\mathbf{x}}$ *and given a path choice* $P : U \to \mathfrak{C}^j(U)$*, we say that* U *is a* P***-set*** *if* U *is path-connected and for every* $\mathbf{x} \in U$, $\gamma_{\mathbf{x}}((0,1]) \subseteq U$*. Moreover, the* ***set of initial points determined by*** P *is defined as*

$$\mathcal{I}_P := \{\gamma_{\mathbf{x}}(0) \in \mathbb{R}^d : \mathbf{x} \in U\}$$

and let us use the notation $\mathbf{a}_{\mathbf{x}} := \gamma_{\mathbf{x}}(0)$ *(see Figure 1).*

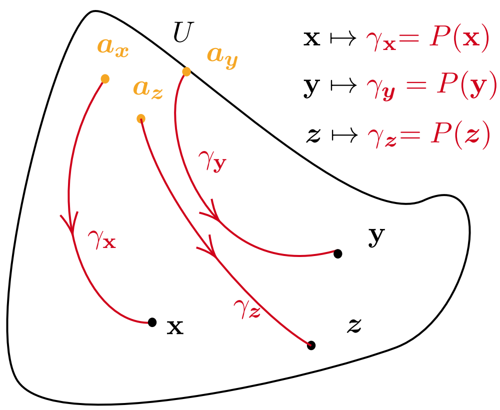

Figure 1: Graphical depiction of the path choice concept

Remark 3.1. *From now on, we assume that for each* $P : U \to \mathfrak{C}^j(U)$ *to be considered,* U *constitutes a* P*-set itself.*

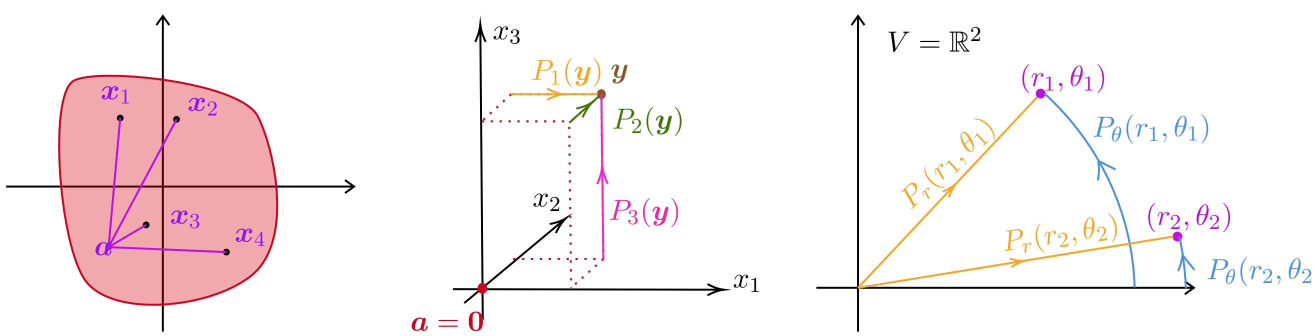

Figure 2: Graphical representations for the path choices discussed in Examples 3.1, 3.2 and 3.3

The geometrical treatment of curves motivates the subsequent notion.

Definition 3.2. *Two path choices* $P : U \to \mathfrak{C}^j(U)$ *and* $Q : U \to \mathfrak{C}^j(U)$ *are said to be* ***equivalent*** *if for every* $\mathbf{x} \in U$ *there exists* $h_{\mathbf{x}} \in \mathrm{Diff}_+([0,1])$ *such that*

$$P(\mathbf{x}) = Q(\mathbf{x}) \circ h_{\mathbf{x}}.$$

After determining a path choice $P : U \to \mathfrak{C}^j(U)$, the following **arc-length function** $s_{(P,\cdot)} : U \times [0,1] \to \mathbb{R}_{\geq 0}$ is induced as the mapping

$$s_{(P,\mathbf{x})}(t) := \int_0^t \left\|\gamma'_{\mathbf{x}}(\tau)\right\| d\tau, \quad t \in [0,1]. \tag{6}$$

In particular, in equation (6), we adopt the special notation for $t = 1$:

$$s_P(\mathbf{x}) := s_{(P,\mathbf{x})}(1). \tag{7}$$

The following result states that equivalent path choices share the same arc-length functions.

Lemma 3.1. *Let $P : U \to \mathfrak{C}^j(U)$ and $Q : U \to \mathfrak{C}^j(U)$ be equivalent path choices. Then for every $\mathbf{x} \in U$ there exists $h_{\mathbf{x}} \in \mathrm{Diff}_+([0,1])$ s.t. $s_{(P,\mathbf{x})} = s_{(Q,\mathbf{x})} \circ h_{\mathbf{x}}$. In particular, for $t = 1$, the total arc-length functions agree and $s_P(\mathbf{x}) = s_Q(\mathbf{x})$.*

Proof. By hypothesis, for every $\mathbf{x} \in U$ there is $h_{\mathbf{x}} \in \mathrm{Diff}_+([0,1])$ such that the identity $P(\mathbf{x})(\tau) = Q(\mathbf{x}) \circ h_{\mathbf{x}}(\tau)$ holds, whence we obtain $P(\mathbf{x})'(\tau) = [Q(\mathbf{x})' \circ h_{\mathbf{x}}(\tau)]h'_{\mathbf{x}}(\tau)$. Taking norms yields

$$\left\|P(\mathbf{x})'(\tau)\right\| = \left\|Q(\mathbf{x})' \circ h_{\mathbf{x}}(\tau)\right\| h'_{\mathbf{x}}(\tau).$$

By integrating this equality from 0 to t, we obtain

$$s_{(P,\mathbf{x})}(t) = \int_0^t \left\|P(\mathbf{x})'(\tau)\right\| d\tau = \int_0^t \left\|Q(\mathbf{x})' \circ h_{\mathbf{x}}(\tau)\right\| h'_{\mathbf{x}}(\tau)\, d\tau.$$

Considering the variable change $u = h_{\mathbf{x}}(\tau)$, we obtain

$$s_{(P,\mathbf{x})}(t) = \int_0^{h_{\mathbf{x}}(t)} \left\|Q(\mathbf{x})'(u)\right\| du = s_{(Q,\mathbf{x})}\big(h_{\mathbf{x}}(t)\big),$$

which proves the first claim. Recalling the notation introduced in (7), and setting $t = 1$ we obtain $s_P(\mathbf{x}) = s_Q(\mathbf{x})$, which completes the proof. □

On the other hand, as a consequence from condition (PC) in Definition 3.1, it follows that for each pair of parameters $t, t' \in [0,1]$, there must exist some $\tau \in [0,1]$ depending on t and t', such that

$$P(\gamma_{\mathbf{x}}(t))(t') = \gamma_{\mathbf{x}}(\tau). \tag{8}$$

In addition, notice that the injectivity of $\gamma_{\mathbf{x}}$ implies that $\gamma_{\mathbf{x}}^{-1} : \mathrm{Im}(\gamma_{\mathbf{x}}) \to [0,1]$ is bijective, whence the map $q_{\mathbf{x},t} := \gamma_{\mathbf{x}}^{-1} \circ P(\gamma_{\mathbf{x}}(t)) : [0,1] \to [0,1]$ constitutes a diffeomorphism, because of the differentiability of $P(\gamma_{\mathbf{x}}(t))$ and $\gamma_{\mathbf{x}}$, alongside with the fact that $\gamma'_{\mathbf{x}} \neq \mathbf{0}$, which by the Inverse Function Theorem implies the differentiability[1] of $\gamma_{\mathbf{x}}^{-1}$ and the claim follows (see Figure 3). In other words, $P(\gamma_{\mathbf{x}}(t)) = \gamma_{\mathbf{x}} \circ q_{\mathbf{x},t}$ and by injectivity, $\tau = q_{\mathbf{x},t}(t')$, whence

$$P(\gamma_{\mathbf{x}}(t))(t') = \gamma_{\mathbf{x}} \circ q_{\mathbf{x},t}(t'). \tag{9}$$

[1] Here we mean *differentiability* in a broader sense, defined over more general types of subsets in $\mathbb{R}^n$, such as the closed sets conformed by $\gamma_{\mathbf{x}}([0,1])$: we say that $f : X \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at a point if it can be locally extended (on a neighborhood of X) to a differentiable function. See, e.g., pages 1 and 2 from [34].

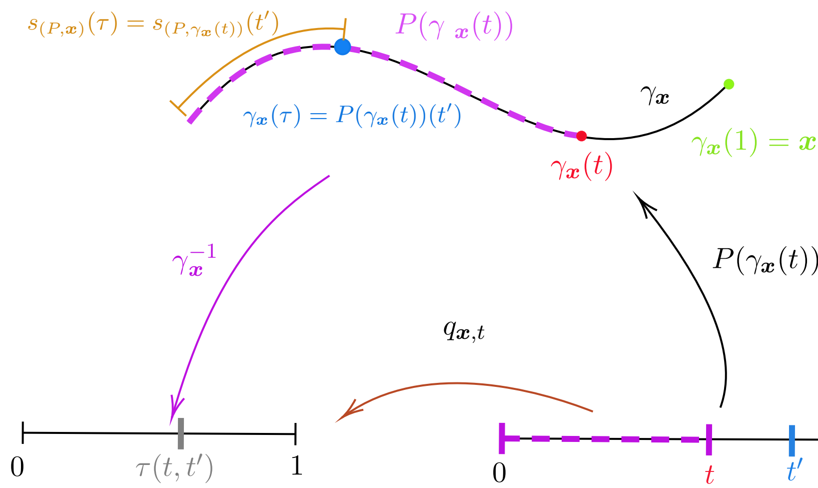

Figure 3: Graphical Analysis of Condition (PC)

What is more, let us define $\widetilde{P}_t : U \to \mathfrak{C}^j(U)$ as $\widetilde{P}_t(\mathbf{x}) := P(\gamma_{\mathbf{x}}(t))$. We have that

$$\begin{aligned} s_{(\widetilde{P}_t,\mathbf{x})}(t') &= \int_0^{t'} \left\| \widetilde{P}_t(\mathbf{x})'(u) \right\| du = \int_0^{t'} \left\| P(\mathbf{x})'(q_{\mathbf{x},\mathbf{t}}(u)) \right\| q'_{\mathbf{x},t}(u) du \\ &= \int_0^{\tau} \left\| P(\mathbf{x})'(v) \right\| dv = s_{(P,\mathbf{x})}(\tau) \quad (v = q_{\mathbf{x},t}(u)) \end{aligned}$$

and in view that $s_{(\widetilde{P}_t,\mathbf{x})}(t') = s_{(P,\gamma_{\mathbf{x}}(t))}(t')$, we have

$$q_{\mathbf{x},t} = s^{-1}_{(P,\mathbf{x})} \circ s_{(P,\gamma_{\mathbf{x}}(t))}, \tag{10}$$

which constitutes an alternative and more explicit expression of the diffeomorphism $q_{\mathbf{x},t}$. The latter fact will be relevant for further results, so let us state it separately.

Proposition 3.1. *Let $P : U \to \mathfrak{C}^j(U)$ be a path choice. It follows that*

$$s_{(P,\gamma_{\mathbf{x}}(t))}(t') = s_{(P,\mathbf{x})}(q_{\mathbf{x},t}(t')), \qquad t' \in [0,1].$$

In particular, when $t' = 1$, $s_P(\gamma_{\mathbf{x}}(t)) = s_{(P,\mathbf{x})}(t)$.

Example 3.1. *Given $\mathbf{a} = (a_1, \dots, a_d) \in \mathbb{R}^d$ with $U \ni \mathbf{a}$ being a star set with respect to $\mathbf{a}$, a simple path choice on U is given by the assignation*

$$\begin{aligned} \overline{P} : U &\to \mathfrak{C}^1(U) \\ \mathbf{x} &\mapsto \overline{P}(\mathbf{x})(t) := \mathbf{a} + (\mathbf{x} - \mathbf{a})t. \end{aligned}$$

In fact, in this case we have $\mathcal{I}_{\overline{P}} = \{\mathbf{a}\}$, $s_{(\overline{P},\mathbf{x})}(t) = t\,\|\mathbf{x} - \mathbf{a}\|$ and $s_{\overline{P}}(\mathbf{x}) = \|\mathbf{x} - \mathbf{a}\|$ (Figure 2). This case and its generalization for any path choice ***from a single point*** *($\mathcal{I}_P = \{\mathbf{a}\}$) were extensively studied in [28, Ch.2].*

Example 3.2. *Let* $U = \prod_{i=1}^{m}[a_i, b_i]$ *for* $2m$ *real numbers* $a_1, \ldots, a_m, b_1, \ldots, b_m$ *such that* $a_i < b_i$*. A simple path choice on* U *is given by the assignation*

$$\begin{aligned} P_i : U &\to \mathfrak{C}^1(U) \\ \mathbf{x} &\mapsto P_i(\mathbf{x})(t) := \mathbf{x} + (a_i - x_i + (x_i - a_i)t)\hat{\mathbf{e}}_i, \end{aligned}$$

for each $i = 1, \ldots, m$*. In fact, in this case we have* $\mathcal{I}_{P_i} = \{a_i\hat{\mathbf{e}}_i\}$ *(for* $m = 3$ *see Figure 2).*

Example 3.3. *Let* $V = \mathbb{R}^2$ *with polar coordinates* (r, θ) *(Figure 2). A pair of path choices on* V *are given by*

$$\begin{aligned} P_\theta : V &\to \mathfrak{C}^j(V) \\ \mathbf{x} &\mapsto P_\theta(r, \theta)(t) := (r, \theta t) \end{aligned}$$

and

$$\begin{aligned} P_r : V &\to \mathfrak{C}^j(V) \\ \mathbf{x} &\mapsto P_r(r, \theta)(t) := (rt, \theta). \end{aligned}$$

Hence, $\mathcal{I}_{P_\theta} = \{(r, 0) : r > 0\}$ *and* $\mathcal{I}_{P_r} = \{(0, \theta) : 0 \leq \theta < 2\pi\} = \{(0, 0)\}$.

To conclude this section, the following useful result provides operational rules for the arc-length function to be used later.

4 Riemann-Liouville Integrals of Type (α, P)

Definition 4.1. *Let* $\alpha > 0$ *and* P *a* j*-th order path choice on* $U \subseteq \mathbb{R}^d$ *for some* $j \in \mathbb{N}$*. For* $f : U \to \mathbb{R}$*, the* ***Riemann-Liouville fractional integral of*** f ***of type*** (α, P) ***at*** $\mathbf{x}$ *is defined as*

$$I_P^\alpha f(\mathbf{x}) := \frac{1}{\Gamma(\alpha)} \int_0^1 [s_P(\mathbf{x}) - s_{(P,\mathbf{x})}(t)]^{\alpha-1} f(\gamma_{\mathbf{x}}(t)) \left\|\gamma_{\mathbf{x}}'(t)\right\| dt. \tag{11}$$

Example 4.1. *When* P *is chosen as in Example 3.2, the fractional integral* $I_{P_i}^\alpha f(\mathbf{x})$ *is precisely the Riemann-Liouville partial integral of* f *of order* α *with respect to the* i*-th variable (e.g. [33, eq. (2.9.3)]):*

$$I_{P_i}^\alpha f(\mathbf{x}) = \frac{1}{\Gamma(\alpha)} \int_{a_i}^{x_i} \frac{f(x_1, \ldots, x_{i-1}, t, x_{i+1}, \ldots, x_n)}{(x_i - t)^{1-\alpha}} dt.$$

Example 4.2. *The integrals of type* (α, P_r) *and* (α, P_θ) *constitute polar analogues of the ones from the last example; let us recall that when we endow* $\mathbb{R}^2$ *with polar coordinates, the arc length of a curve* $[a, b] \ni t \mapsto (r(t), \theta(t))$ *is given by the integral*

$$\int_a^b \sqrt{r'(t)^2 + r(t)^2\theta'(t)^2}\, dt,$$

as the metric in this case is given by the differential form $dr^2 + r\, d\theta^2$*. Consequently,* $I_{P_r}^\alpha f(r, \theta) = r^\alpha I_{0+}^\alpha f(rt, \theta)\big|_{t=1}$ *and* $I_{P_\theta}^\alpha f(r, \theta) = (r\theta)^\alpha I_{0+}^\alpha f(r, \theta t)\big|_{t=1}$.

Example 4.3. *As the reader may expect, a similar treatment could be carried out in spherical and cylindrical coordinates: for the first case we should consider the metric* $dr^2 + r\ d\theta^2 + dz^2$ *and the path choices* $P_r(r,\theta,z)(t) = (rt,\theta,z)$, $P_\theta(r,\theta,z)(t) = (r,\theta t,z)$ *and* $P_z(r,\theta,z) = (r,\theta,zt)$. *Analogously, for the spherical setting the metric* $d\rho^2 + \rho\ d\theta^2 + \rho\sin\theta\ d\varphi^2$ *serves for the arc lengths and suggested path choices are* $P_\rho(\rho,\theta,\varphi)(t) = (\rho t,\theta,\varphi)$, $P_\theta(\rho,\theta,\varphi)(t) = (\rho,\theta t,\varphi)$ *and* $P_\varphi(\rho,\theta,\varphi)(t) = (\rho,\theta,\varphi t)$. *In turn, we get the following integral operators:*

$$\begin{aligned} I^\alpha_{P_r} f(r,\theta,z) &= r^\alpha I^\alpha_{0+} f(rt,\theta,z)|_{t=1}, & I^\alpha_{P_\theta} f(r,\theta,z) &= (r\theta)^\alpha I^\alpha_{0+} f(r,\theta t,z)|_{t=1}, \\ I^\alpha_{P_z} f(r,\theta,z) &= z^\alpha I^\alpha_{0+} f(r,\theta,zt)|_{t=1}, & I^\alpha_{P_\rho}(\rho t,\theta,\varphi) &= \rho^\alpha I^\alpha_{0+} f(\rho t,\theta,\varphi)|_{t=1}, \\ I^\alpha_{P_\theta}(\rho,\theta t,\varphi) &= (\rho\theta)^\alpha I^\alpha_{0+} f(\rho,\theta t,\varphi)|_{t=1}, & I^\alpha_{P_\varphi}(\rho,\theta,\varphi t) &= (\rho\sin\theta)^\alpha I^\alpha_{0+} f(\rho,\theta,\varphi t)|_{t=1}. \end{aligned}$$

Example 4.4. *Additionally, equation* (11) *generalizes fractional directional integrals defined over straight segments (Example 3.1) as in [26, eq. (14)]. Specifically,*

$$I^\alpha_P f(\mathbf{x}) = \|\mathbf{x} - \mathbf{a}\|^\alpha I^\alpha_{0+} f(\mathbf{a} + (\mathbf{x} - \mathbf{a})t)|_{t=1},$$

where the integral term in the last equation is precisely the definition proposed in [26].

Just as it happened with the arc-length function $s_{(P,\mathbf{x})}(t)$, the following result shows that this integral does not depend on the path choice function.

Proposition 4.1. *For two equivalent path choices* $P, Q : U \to \mathfrak{C}^j(U)$ *we have* $I^\alpha_P = I^\alpha_Q$ *for* $\alpha > 0$.

Proof. By (11) and Lemma 3.1 we have that

$$\begin{aligned} I^\alpha_P f(\mathbf{x}) &= \frac{1}{\Gamma(\alpha)} \int_0^1 \left[s_P(\mathbf{x}) - s_{(P,\mathbf{x})}(t)\right]^{\alpha-1} f\big(\gamma_{P,\mathbf{x}}(t)\big) \left\|\gamma'_{P,\mathbf{x}}(t)\right\| dt \\ &= \frac{1}{\Gamma(\alpha)} \int_0^1 \left[s_Q(\mathbf{x}) - s_{(Q,\mathbf{x})}\big(h_\mathbf{x}(t)\big)\right]^{\alpha-1} f\big(\gamma_{Q,\mathbf{x}}(h_\mathbf{x}(t))\big) \left\|\gamma'_{Q,\mathbf{x}}(h_\mathbf{x}(t))\right\| h'_\mathbf{x}(t)\, dt. \end{aligned}$$

Now let $u = h_\mathbf{x}(t)$. Then

$$I^\alpha_P f(\mathbf{x}) = \frac{1}{\Gamma(\alpha)} \int_0^1 \left[s_Q(\mathbf{x}) - s_{(Q,\mathbf{x})}(u)\right]^{\alpha-1} f\big(\gamma_{Q,\mathbf{x}}(u)\big) \left\|\gamma'_{Q,\mathbf{x}}(u)\right\| du = I^\alpha_Q f(\mathbf{x}).$$

Since the previous identity holds for every $\mathbf{x} \in U$, we conclude that $I^\alpha_P f = I^\alpha_Q f$ on U. □

As pointed out in [35], in analogy with the classical Riemann–Liouville fractional integrals, we may regard $I^\alpha_P f$ as integrals of $f \circ \gamma_\mathbf{x}$ with respect to the Lebesgue–Stieltjes measure

$$\mu_{P,\alpha,\mathbf{x}} := -\frac{(s_P(\mathbf{x}) - s_{(P,\mathbf{x})}(\cdot))^\alpha}{\Gamma(\alpha+1)},$$

induced by α and P, for each $\mathbf{x} \in U$. Therefore, in the context of L^p spaces defined on abstract measure spaces, we propose the following extension.

Definition 4.2. *For* $\alpha > 0$, $U \subset \mathbb{R}^d$, *a path choice* $P : U \to \mathfrak{C}^j(U)$ *and* $1 \le p < \infty$, *let us define*

$$L^p_{(\alpha,P)}(U) := \left\{f : U \to \mathbb{R} : f \circ \gamma_\mathbf{x} \in L^p([0,1]; \mu_{\alpha,P,\mathbf{x}}),\ \forall\ \mathbf{x} \in U\right\},$$

endowed with the norm

$$\|f\|_{L^p_{(\alpha,P)}} := \sup_{\mathbf{x}\in U} \left\{(I^\alpha_P |f|^p(\mathbf{x}))^{1/p}\right\}.$$

Notice that $I_P^\alpha f$ exists on U as long as $f \in L^1_{(\alpha,P)}(U)$, so the latter space is the biggest one such that $I_P^\alpha f$ exists. Actually, it is immediate to reinforce this fact.

Proposition 4.2. *Let* $1 \leq p_1 \leq p_2 \leq \infty$. *Then* $L^{p_2}_{(\alpha,P)}(U) \subseteq L^{p_1}_{(\alpha,P)}(U)$.

Proof. Using [36, Corollary 2, p. 397], we have $\left\| f\mu'_{\alpha,P,\mathbf{x}} \right\|_{L^{p_1}} \leq c_{\mathbf{x}} \left\| f\mu'_{\alpha,P,\mathbf{x}} \right\|_{L^{p_2}}$ with $c_{\mathbf{x}}$ defined as

$$c_{\mathbf{x}} := \begin{cases} \left(\frac{s_P(\mathbf{x})^\alpha}{\Gamma(\alpha+1)}\right)^{\frac{p_2-p_1}{p_1 p_2}} & \text{if } p_2 < \infty \text{ and} \\ \left(\frac{s_P(\mathbf{x})^\alpha}{\Gamma(\alpha+1)}\right)^{\frac{1}{p_1}} & \text{if } p_2 = \infty. \end{cases}$$

Taking supremes over every $\mathbf{x} \in U$ we achieve the desired result. □

Another way of expressing I_P^α consists of defining the **pre-processing operator** $\mathcal{Q}_1 : L^1_{(\alpha,P)}(U) \to L^1([0, s_P(\mathbf{x})], \mu_{\alpha,P,\mathbf{x}})$ and the **post-processing operator** $\mathcal{Q}_2 : L^1([0, s_P(\mathbf{x})], \mu_{\alpha,P,\mathbf{x}}) \to L^1_{(\alpha,P)}(U)$ as

$$(\mathcal{Q}_1\, f(\mathbf{x}))(\sigma) = f \circ \gamma_{\mathbf{x}} \circ s^{-1}_{(P,\mathbf{x})}(\sigma) \text{ and } (\mathcal{Q}_2\, g)(\mathbf{x}) = g \circ s_P(\mathbf{x}), \tag{12}$$

respectively. It is straightforward to verify that Q_2 is the inverse of Q_1. Indeed, for $g \in L^1([0,1], \mu_{\alpha,P,\mathbf{x}})$ we compute

$$(\mathcal{Q}_2\, g)(\mathbf{x}) = g\left(s_{(P,\mathbf{x})}(1)\right)$$

$$\Big(\mathcal{Q}_1(\mathcal{Q}_2\, g)(\mathbf{x})\Big)(\sigma) = g\left(s_{(P,\gamma_{\mathbf{x}} \circ s^{-1}_{(P,\mathbf{x})}(\sigma))}(1)\right)$$

By Proposition 3.1, if we consider $t = s^{-1}_{(P,\mathbf{x})}(\sigma)$ we have

$$s_{\left(P,\gamma_{\mathbf{x}} \circ s^{-1}_{(P,\mathbf{x})}(\sigma)\right)}(1) = s_{(P,\gamma_{\mathbf{x}}(t))}(1) = s_{(P,\mathbf{x})}(t) = \sigma,$$

and therefore $\Big(\mathcal{Q}_1(\mathcal{Q}_2\, g)(\mathbf{x})\Big)(\sigma) = g(\sigma)$. On the other hand,

$$\mathcal{Q}_2(\mathcal{Q}_1 f)(\sigma)(\mathbf{x}) = f(\gamma_{\mathbf{x}}(s^{-1}_{(P,\mathbf{x})}(s_P(\mathbf{x})))) = f(\gamma_{\mathbf{x}}(1)) = f(\mathbf{x}).$$

In this way, by unifying the notation of the operators in equation (12), let $\mathcal{Q}_P := \mathcal{Q}_1$, whence $\mathcal{Q}_2 = \mathcal{Q}_P^{-1}$ and note that

$$I_P^\alpha f(\mathbf{x}) = \frac{1}{\Gamma(\alpha)} \int_0^{s_P(\mathbf{x})} (s_P(\mathbf{x}) - \sigma)^{\alpha-1} f(s^{-1}_{(P,\mathbf{x})}(\sigma)) d\sigma,$$

it follows that the equivalent representation of equation (11) is

$$I_P^\alpha f(\mathbf{x}) = \mathcal{Q}_P^{-1} \circ I^\alpha_{0+} \circ \mathcal{Q}_P f(\mathbf{x}). \tag{13}$$

The extension proposed in Definition 4.1 also satisfies a semigroup-like property, which simultaneously extends the corresponding property of the classical Riemann-Liouville integrals.

Proposition 4.3. *Let* $f \in L^1_{(\alpha,P)}(U)$ *with* α, $\beta > 0$ *and a path choice* $P : U \to \mathfrak{C}^1(U)$.

a) $\lim_{\alpha \to 0^+} I_P^\alpha f(\mathbf{x}) = f(\mathbf{x})$.

b) Let $f \in L^1_{(1,P)}(U)$. Then

$$I^{\alpha}_{P}\big(I^{\beta}_{P}f\big)(\mathbf{x}) = I^{\alpha+\beta}_{P}f(\mathbf{x}), \quad \forall \mathbf{x} \in U. \tag{14}$$

Proof. For a) we can use (13) alongside with the property:

$$\lim_{\alpha\to 0^+} I^{\alpha}_{a+}f(x) = f(x)$$

(cf. [4, Theorem 2.7]) and b) follows from (13), together with [37, Theorem 2.2]. □

Remark 4.1. *Analogously to [27, Remark 2.1], the previous result allows us to adopt the convention $I^0_P f(\mathbf{x}) := f(\mathbf{x})$.*

Another ordering result is given in terms of the integration orders.

Proposition 4.4. *Let $0 \le \beta < \alpha$, $P : U \to \mathfrak{C}^j(U)$ and assume that $\mathbf{x} \mapsto s_P(\mathbf{x})$ is bounded on U. For each $q \ge 1$, it holds that $L^q_{(\beta,P)}(U) \subseteq L^q_{(\alpha,P)}(U)$.*

Proof. Let $f \in L^q_{(\beta,P)}(U)$. Due to the semigroup property,

$$I^{\alpha}_{P}|f|^q(\mathbf{x}) = I^{\alpha-\beta}_{P} I^{\beta}_{P}|f|^q(\mathbf{x}) \le \frac{s_P(\mathbf{x})^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\|f\|^q_{L^q_{(\beta,P)}}.$$

Now it suffices to take q-th roots and supremums over $\mathbf{x} \in U$ to conclude that $f \in L^q_{(\alpha,P)}(U)$, since $s_P(\mathbf{x})^{\alpha-\beta}$ is bounded and $\alpha - \beta > 0$ by hypothesis. □

Corollary 4.1. *$L^1_{(\beta,P)} \subseteq L^1_{(1,P)}(U) \subseteq L^1_{(\alpha,P)}(U)$ for any $0 \le \beta < 1$ and $\alpha > 1$.*

Before continuing, let us introduce the subsequent analogues of the C^n spaces.

Definition 4.3. *We define the* ***space of*** (α, P)***-continuous functions on*** U ***as***

$$C_{(\alpha,P)}(U) := \{f : U \to \mathbb{R} : f \circ \gamma_{\mathbf{x}}\mu'_{\alpha,P,\mathbf{x}} \in C[0,1],\ \forall\ \mathbf{x} \in U\},$$

endowed with the norm

$$\|f\|_{C_{(\alpha,P)}} := \sup_{\mathbf{x}\in U}\left\{\left\|f \circ \gamma_{\mathbf{x}}\mu'_{\alpha,P,\mathbf{x}}\right\|_{C([0,1])}\right\}.$$

Additionally, we naturally define the ***space of*** (α, P)***-continuous functions of order*** n *as*

$$C^n_{(\alpha,P)}(U) := \{f : U \to \mathbb{R} : f \circ \gamma_{\mathbf{x}} \in C^{n-1}([0,1]) \text{ and } f \circ \gamma_{\mathbf{x}}\mu'_{\alpha,P,\mathbf{x}} \in C^n([0,1]),\ \forall\ \mathbf{x} \in U\},$$

equipped with the norm

$$\|f\|_{C^n_{(\alpha,P)}} := \sup_{\mathbf{x}\in U}\left\{\|f \circ \gamma_{\mathbf{x}}\|_{C^{n-1}([0,1])} + \left\|\left(f \circ \gamma_{\mathbf{x}}\mu'_{\alpha,P,\mathbf{x}}\right)^{(n)}\right\|_{C([0,1])}\right\}$$

where $\|\cdot\|_{C^j([a,b])}$ is commonly defined as $||g||_{C^j([a,b])} = \sum_{i=j}^{k}\left\|g^{(i)}\right\|_{C([a,b])}$ (see, e.g., [38, p. 214]). In addition, let us adopt the convention $C^0_{(\alpha,P)}(U) := C_{(\alpha,P)}(U)$.

To conclude this section, we observe that the fractional integral along paths acts in a simple and explicit way on powers of the arc-length function. More precisely, for $\alpha > 0$ we obtain

$$I_P^\alpha s_P(\mathbf{x})^\beta \;=\; \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)}\, s_P(\mathbf{x})^{\alpha+\beta}, \quad \text{with } \beta > -1. \tag{15}$$

Moreover, this observation can be extended to show that the computation of $I_P^\alpha f$ may be reduced to a one-variable fractional integral, whenever the function to be integrated can be expressed in terms of the corresponding arc-length function $s_P(\mathbf{x})$.

Proposition 4.5. *Let $\alpha, k > 0$ and consider a path choice $P : U \to \mathfrak{C}^1(U)$. Suppose that for $f \in L^1_{(\alpha,P)}(U)$ there exists $g : \mathbb{R}_{\geq 0} \to \mathbb{R}$ such that $f(\mathbf{x}) = g(s_P(\mathbf{x}))$. Then*

$$I_P^\alpha f(\mathbf{x}) = (I_{0+}^\alpha g)(s_P(\mathbf{x})). \tag{16}$$

Proof. By definition and Proposition 3.1, we obtain

$$\begin{aligned} I_P^\alpha f(\mathbf{x}) &= \frac{1}{\Gamma(\alpha)} \int_0^1 [s_P(\mathbf{x}) - s_{(P,\mathbf{x})}(t)]^{\alpha-1} g(s_P(\gamma_{\mathbf{x}}(t))) \big\|\gamma'_{\mathbf{x}}(t)\big\| dt \\ &= \frac{1}{\Gamma(\alpha)} \int_0^1 [s_P(\mathbf{x}) - s_{(P,\mathbf{x})}(t)]^{\alpha-1} g(s_{(P,\mathbf{x})}(t)) \big\|\gamma'_{\mathbf{x}}(t)\big\| dt. \end{aligned}$$

Then we apply the variable change $\sigma = s_{(P,\mathbf{x})}(t)$, $d\sigma = \|\gamma'_{\mathbf{x}}(t)\| dt$ and we obtain

$$I_P^\alpha f(\mathbf{x}) = \frac{1}{\Gamma(\alpha)} \int_0^{s_P(\mathbf{x})} (s_P(\mathbf{x}) - \sigma)^{\alpha-1} g(\sigma) d\sigma = (I_{0+}^\alpha g)(s_P(\mathbf{x})).$$

□

5 Riemann-Liouville Differentials of type (α, P)

The following notion is a direct extension of the total differentials of order $\alpha > 0$ along a given trajectory, previously proposed in [27].

Definition 5.1. *Take $\alpha > 0$, $\alpha \notin \mathbb{N}$, $n \in \mathbb{N}$ such that $n - 1 < \alpha < n$ and some path choice $P : U \to \mathfrak{C}^j(U)$. For $f : U \to \mathbb{R}$, the **Riemann-Liouville total differential of type** (α, P) **of** f **at** $\mathbf{x}$ is given by*

$$D_P^\alpha f(\mathbf{x})(\mathbf{v}_1, \ldots, \mathbf{v}_n) := D^n I_P^{n-\alpha} f(\mathbf{x})(\mathbf{v}_1, \ldots, \mathbf{v}_n). \tag{17}$$

We say that f is (α, P)-differentiable at $\mathbf{x}$ if $D_P^\alpha f(\mathbf{x})$ exists and we express the set of (α, P)-differentiable functions in U as $D_{(\alpha,P)}(U)$. In particular, for $\alpha \in (0, 1]$, equation (17) reduces to the computation of a gradient (e.g. [39, Theorem 8.5]):

$$D_P^\alpha f(\mathbf{x})(\mathbf{v}) = \langle \nabla I_P^{1-\alpha} f(\mathbf{x}), \mathbf{v} \rangle. \tag{18}$$

We have the following *first fundamental theorem* for the operators D_P^α. Before that, let us introduce some notation:

$$\mathbf{v}_{\mathbf{x}} := \gamma'_{\mathbf{x}}(1), \quad \mathbf{u}_{\mathbf{x}} := \frac{\mathbf{v}_{\mathbf{x}}}{\|\mathbf{v}_{\mathbf{x}}\|} \text{ and } (\mathbf{u}_{\mathbf{x}})^n := \underbrace{(\mathbf{u}_{\mathbf{x}}, \mathbf{u}_{\mathbf{x}}, \ldots, \mathbf{u}_{\mathbf{x}})}_{n-\text{tuple}}.$$

Theorem 5.1. *Let $f : U \to \mathbb{R}$ and a path choice $P : U \to \mathfrak{C}^j(U)$ such that $I_P^{n-\alpha} f \in D_{(\alpha,P)}(U)$ with $n \in \mathbb{N}$ such that $n-1 < \alpha \leq n$ with $\alpha > 0$. We have that*

$$D_P^{\alpha} I_P^{\alpha} f(\mathbf{x}) \, (\mathbf{u_x})^n = f(\mathbf{x}). \tag{19}$$

Proof. We first prove the result for the case $\alpha = n = 1$. From equation (18) and the classical multivariable chain rule (see, e.g., [40, Theorem 8.8]), we have

$$\frac{\partial I_P^1 f(\gamma_{\mathbf{x}}(t))}{\partial t} = \langle \nabla I_P^1 f(\gamma_{\mathbf{x}}(t)), \, \gamma'_{\mathbf{x}}(t) \rangle.$$

Evaluating at $t = 1$, it follows that

$$\frac{\partial I_P^1 f(\gamma_{\mathbf{x}}(t))}{\partial t}\Big|_{t=1} = \langle \nabla I_P^1 f(\mathbf{x}), \, \mathbf{v_x} \rangle = D^1 I_P^1 f(\mathbf{x})(\mathbf{v_x}). \tag{20}$$

On the other hand, by the classical Fundamental Theorem of Calculus and (9),

$$\begin{aligned}
\frac{\partial I_P^1 f(\gamma_{\mathbf{x}}(t))}{\partial t} &= \frac{\partial}{\partial t} \int_0^1 f(P(\gamma_{\mathbf{x}}(t))(\tau)) \, \|P(\gamma_{\mathbf{x}}(t))'(\tau)\| \, d\tau \\
&= \frac{\partial}{\partial t} \int_0^1 f(\gamma_{\mathbf{x}}(q_{\mathbf{x},t}(\tau))) \, \|\gamma'_{\mathbf{x}}(q_{\mathbf{x},t}(\tau))\| \, q_{\mathbf{x},t}(\tau) d\tau \quad (u = q_{\mathbf{x},t}(\tau)) \\
&= \frac{\partial}{\partial t} \int_0^t f(\gamma_{\mathbf{x}}(u)) \, \|\gamma'_{\mathbf{x}}(u)\| \, du \\
&= f(\gamma_{\mathbf{x}}(t)) \, \|\gamma'_{\mathbf{x}}(t)\|.
\end{aligned}$$

Therefore,

$$\frac{\partial I_P^1 f(\gamma_{\mathbf{x}}(t))}{\partial t}\Big|_{t=1} = f(\mathbf{x}) \, \|\mathbf{v_x}\|. \tag{21}$$

Combining both expressions (20) and (21), we obtain the desired result for $\alpha = 1$. For the case $\alpha = n$, we proceed by induction. The base case $n = 1$ has already been proved. Assume the statement holds for some $n \geq 1$; that is, $D_P^n I_P^n f(\mathbf{x})(\mathbf{u_x})^n = f(\mathbf{x})$. Then, for $n+1$ we have

$$\begin{aligned}
D_P^{n+1} I_P^{n+1} f(\mathbf{x})(\mathbf{u_x})^{n+1} &= D_P^1 \big(D_P^n I_P^n \big(I_P^1 f \big) (\mathbf{x})(\mathbf{u_x})^n \big) (\mathbf{u_x}) \\
&= D_P^1 \big(I_P^1 f(\mathbf{x}) \big) (\mathbf{u_x}) \\
&= f(\mathbf{x}),
\end{aligned}$$

which completes the induction and proves the result for $\alpha = n$.

For an arbitrary $\alpha > 0$, it suffices to apply the integer-order case together with Proposition 4.3 (b). Indeed, let $n \in \mathbb{N}$ such that $n - 1 < \alpha < n$, then

$$\begin{aligned}
D_P^{\alpha} I_P^{\alpha} f(\mathbf{x})(\mathbf{u_x})^n &= D_P^n I_P^{n-\alpha} I_P^{\alpha} f(\mathbf{x})(\mathbf{u_x})^n \\
&= D_P^n I_P^n f(\mathbf{x})(\mathbf{u_x})^n \\
&= f(\mathbf{x}).
\end{aligned}$$

This concludes the proof. □

The following result will be important to state our main second theorem. Geometrically speaking, it tells us that the maximal direction of change of the scalar field $\mathbf{x} \mapsto s_P(\mathbf{x})$ is precisely the tangent direction defined by P at each $\mathbf{x} \in U$.

Lemma 5.1. *For any path choice $P : U \to \mathfrak{C}^1(U)$ such that s_P is differentiable in (the interior of) U we have*

$$D^1 s_P(\mathbf{x})(\mathbf{u_x}) = 1. \tag{22}$$

Proof. The result follows from definition and the classical chain rule:

$$D^1 s_P(\mathbf{x})(\mathbf{u_x}) = \frac{1}{\|\gamma'_{\mathbf{x}}(1)\|} \sum_{j=1}^{m} \frac{\partial s_{P,\mathbf{x}}(t)}{\partial xj}\bigg|_{t=1} \frac{\mathrm{d}\gamma^i_{\mathbf{x}}(t)}{\mathrm{d}t}\bigg|_{t=1} = \frac{1}{\|\gamma'_{\mathbf{x}}(1)\|} \frac{\mathrm{d}s_P(\gamma_{\mathbf{x}}(t))}{\mathrm{d}t}\bigg|_{t=1} = 1.$$

□

Conversely, we have the following result concerning the inverse composition of integrals and differentials of type (α, P); it is a *second fundamental theorem* for D^α_P. As we did previously, let us introduce auxiliary notation:

$$\mathbf{w_x} := \frac{\gamma'_{\mathbf{x}}(0)}{\|\gamma'_{\mathbf{x}}(0)\|}.$$

Theorem 5.2. *Let $\alpha > 0$, $n \in \mathbb{N}$ with $n - 1 < \alpha \le n$ and $P : U \to \mathfrak{C}^1(U)$. If $f : U \to \mathbb{R}$ satisfies $D^\alpha_P f(\cdot)(\mathbf{u}_{(\cdot)})^n \in C^n_{(\alpha,P)}(U)$, then the following identity holds:*

$$I^\alpha_P D^\alpha_P f(\mathbf{x})(\mathbf{u_x})^n = f(\mathbf{x}) - \sum_{j=1}^{n} \frac{D^{\alpha-j}_P f(\mathbf{a_x})(\mathbf{w_x})^{n-j}}{\Gamma(\alpha - j + 1)} s_P(\mathbf{x})^{\alpha-j}. \tag{23}$$

Proof. We follow the line of reasoning from the previous theorem. First, for $\alpha = 1$ we appeal to the second fundamental theorem of calculus (e.g. [40, Theorem 10.2]), which allows us to prove the theorem for this scenario:

$$\begin{aligned}
I^1_P D^1 f(\mathbf{x})(\mathbf{u}_x) &= \int_0^1 D^1 f(\gamma_{\mathbf{x}}(t))(\mathbf{u}_{\gamma_{\mathbf{x}}(t)}) \big\|\gamma'_{\mathbf{x}}(t)\big\| dt \\
&= \int_0^1 D^1 f(\gamma_{\mathbf{x}}(t))(\gamma'_{\mathbf{x}}(t)) dt \\
&= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} f(\gamma_{\mathbf{x}}(t)) \\
&= f(\gamma_{\mathbf{x}}(1)) - f(\gamma_{\mathbf{x}}(0)) \\
&= f(\mathbf{x}) - f(\mathbf{a_x}).
\end{aligned}$$

For $\alpha = n$, we proceed via induction and assume the result is valid for $n \ge 1$. Then, for $n + 1$ we see that

$$\begin{aligned}
I^{n+1}_P D^{n+1} f(\mathbf{x})(\mathbf{u_x})^{n+1} &= I^1_P I^n_P D^n \left(D^1 f(\mathbf{x})(\mathbf{u_x})\right)(\mathbf{u_x})^n \\
&= I^1_P \left[D^1 f(\mathbf{x})(\mathbf{u_x}) - \sum_{j=1}^{n} \frac{D^{n-j+1} f(\mathbf{a_x})(\mathbf{w_x})^{n-j+1}}{\Gamma(n-j+1)} s_P(\mathbf{x})^{n-j} \right] \\
&= f(\mathbf{x}) - f(\mathbf{a_x}) - \sum_{j=1}^{n} \frac{I^1_P \left(D^{n-j+1} f(\mathbf{a_x})(\mathbf{w_x})^{n-j+1} s_P(\mathbf{x})^{n-j}\right)}{\Gamma(n-j+1)}.
\end{aligned}$$

Now, let us compute the integral of type $(1, P)$ in the summand. For such purpose, we resort to Proposition 3.1, but let us notice that due to (9) we also have,

$$\mathbf{a}_{\gamma_{\mathbf{x}}(t)} = P(\gamma_{\mathbf{x}}(t))(0) = \gamma_{\mathbf{x}}(q_{\mathbf{x},t}(0)) = \mathbf{a_x},$$

and

$$\mathbf{w}_{\gamma_{\mathbf{x}}(t)} = \frac{P(\gamma_{\mathbf{x}}(t))'(\tau)|_{\tau=0}}{\|P(\gamma_{\mathbf{x}}(t))'(\tau)\||_{\tau=0}} = \frac{\gamma'_{\mathbf{x}}(q_{\mathbf{x},t}(\tau))|_{\tau=0}}{\|\gamma'_{\mathbf{x}}(q_{\mathbf{x},t}(\tau))\||_{\tau=0}} = \mathbf{w}_{\mathbf{x}}.$$

Therefore

$$\begin{aligned} I_P^1\left(D^{n-j+1}f(\mathbf{a}_{\mathbf{x}})(\mathbf{w}_{\mathbf{x}})^{n-j+1}s_P(\mathbf{x})^{n-j}\right) &= \int_0^1 D^{n-j+1}f(\mathbf{a}_{\gamma_{\mathbf{x}}(t)})(\mathbf{w}_{\gamma_{\mathbf{x}}(t)})^{n-j+1}s_P(\gamma_{\mathbf{x}}(t))^{n-j}\left\|\gamma'_{\mathbf{x}}(t)\right\|dt \\ &= \int_0^1 D^{n-j+1}f(\mathbf{a}_{\mathbf{x}})\left(\mathbf{w}_{\mathbf{x}}\right)^{n-j+1} s_{(P,\mathbf{x})}(t)^{n-j}\left\|\gamma'_{\mathbf{x}}(t)\right\|dt \\ &= D^{n-j+1}f(\mathbf{a}_{\mathbf{x}})(\mathbf{w}_{\mathbf{x}})^{n-j+1}\int_0^1 s_{(P,\mathbf{x})}(t)^{n-j}\left\|\gamma'_{\mathbf{x}}(t)\right\|dt \\ &= D^{n-j+1}f(\mathbf{a}_{\mathbf{x}})(\mathbf{w}_{\mathbf{x}})^{n-j+1}I_P^1 s_P(\mathbf{x})^{n-j} \\ &= D^{n-j+1}f(\mathbf{a}_{\mathbf{x}})(\mathbf{w}_{\mathbf{x}})^{n-j+1}\frac{\Gamma(n-j+1)}{\Gamma(n-j+2)}s_P(\mathbf{x})^{n-j+1}, \end{aligned}$$

where the last equality follows from (15) for $\alpha = 1$ and $\beta = n - j$. Therefore, continuing with the foregoing computations,

$$\begin{aligned} I_P^{n+1}D^{n+1}f(\mathbf{x})(\mathbf{u}_{\mathbf{x}})^{n+1} &= f(\mathbf{x}) - f(\mathbf{a}_{\mathbf{x}}) - \sum_{j=1}^{n}\frac{D^{n-j+1}f(\mathbf{a}_{\mathbf{x}})(\mathbf{w}_{\mathbf{x}})^{n-j+1}s_P(\mathbf{x})^{n-j+1}}{\Gamma(n-j+2)} \\ &= f(\mathbf{x}) - \sum_{j=1}^{n+1}\frac{D^{n-j+1}f(\mathbf{a}_{\mathbf{x}})(\mathbf{w}_{\mathbf{x}})^{n+1-j}s_P(\mathbf{x})^{n+1-j}}{\Gamma(n-j+2)} \end{aligned}$$

and the result follows in this case. Moving forward, for $\alpha > 0$ and $n \in \mathbb{N}$ with $n-1 < \alpha \leq n$ we see that applying the integer-order case to the function $I_P^{n-\alpha}f(\mathbf{x})$ we obtain

$$\begin{aligned} I_P^n D^n I_P^{n-\alpha}f(\mathbf{x})(\mathbf{u}_{\mathbf{x}})^n &= I_P^{n-\alpha}f(\mathbf{x}) - \sum_{j=1}^{n}\frac{D^{n-j}I_P^{n-\alpha}f(\mathbf{a}_{\mathbf{x}})(\mathbf{w}_{\mathbf{x}})^{n-j}}{\Gamma(n-j+1)}s_P(\mathbf{x})^{n-j} \\ &= I_P^{n-\alpha}f(\mathbf{x}) - \sum_{j=1}^{n}\frac{D_P^{\alpha-j}f(\mathbf{a}_{\mathbf{x}})(\mathbf{w}_{\mathbf{x}})^{n-j}}{\Gamma(n-j+1)}s_P(\mathbf{x})^{n-j}. \end{aligned}$$

Now let us employ Proposition 4.3 (b), it follows that

$$I_P^{n-\alpha}\left(I_P^{\alpha}D_P^{\alpha}f(\mathbf{x})(\mathbf{u}_{\mathbf{x}})^n - f(\mathbf{x})\right) = \sum_{j=1}^{n}\frac{D_P^{\alpha-j}f(\mathbf{a}_{\mathbf{x}})(\mathbf{w}_{\mathbf{x}})^{n-j}}{\Gamma(n-j+1)}s_P(\mathbf{x})^{n-j}.$$

Thus, using Theorem 5.1 and Lemma 5.1 we deduce that

$$\begin{aligned} I_P^{\alpha}D_P^{\alpha}f(\mathbf{x})(\mathbf{u}_{\mathbf{x}})^n - f(\mathbf{x}) &= \sum_{j=1}^{n}\frac{D_P^{\alpha-j}f(\mathbf{a}_{\mathbf{x}})(\mathbf{w}_{\mathbf{x}})^{n-j}}{\Gamma(n-j+1)}D_P^{n-\alpha}s_P(\mathbf{x})^{n-j}(\mathbf{u}_{\mathbf{x}}) \\ &= \sum_{j=1}^{n}\frac{D_P^{\alpha-j}f(\mathbf{a}_{\mathbf{x}})(\mathbf{w}_{\mathbf{x}})^{n-j}}{\Gamma(n-j+1)}D^1 I_P^{1-n+\alpha}s_P(\mathbf{x})^{n-j}(\mathbf{u}_{\mathbf{x}}) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{n} \frac{D_P^{\alpha-j} f(\mathbf{a_x})(\mathbf{w_x})^{n-j}}{\Gamma(\alpha-j+2)} D^1 s_P(\mathbf{x})^{\alpha-j+1}(\mathbf{u_x}).\\
&= \sum_{j=1}^{n} \frac{D_P^{\alpha-j} f(\mathbf{a_x})(\mathbf{w_x})^{n-j}}{\Gamma(\alpha-j+1)} s_P(\mathbf{x})^{\alpha-j} D^1 s_P(\mathbf{x})(\mathbf{u_x})\\
&= \sum_{j=1}^{n} \frac{D_P^{\alpha-j} f(\mathbf{a_x})(\mathbf{w_x})^{n-j}}{\Gamma(\alpha-j+1)} s_P(\mathbf{x})^{\alpha-j}.
\end{aligned}
$$

□

6 Riemann-Liouville Operators of Type (α, k, P)

We now proceed to generalize the definition of the Riemann–Liouville fractional integral of f of type (α, P) by perturbing the order α via an additional parameter $k \in \mathbb{R}_{>0}$. Accordingly, we introduce the following definition.

Definition 6.1. *Let $\alpha, k > 0$ and P a j-th order path choice on $U \subseteq \mathbb{R}^d$ for some $j \in \mathbb{N}$. For $f : U \to \mathbb{R}$, the **Riemann-Liouville fractional integral of f of type** (α, k, P) **at $\mathbf{x}$** is defined as*

$$
I_P^{(\alpha,k)} f(\mathbf{x}) := \frac{1}{k\Gamma_k(\alpha)} \int_0^1 [s_P(\mathbf{x}) - s_{(P,\mathbf{x})}(t)]^{\frac{\alpha}{k}-1} f(\gamma_{\mathbf{x}}(t)) \big\|\gamma'_{\mathbf{x}}(t)\big\| dt. \tag{24}
$$

Observe that, according to equations (24) and (11), the operator $I_P^{(\alpha,k)} \to I_P^{\alpha}$ (pointwise) when $k \to 1$.

Remark 6.1. *From Definition 6.1 and Proposition 2.1 it may be obtained the following functional relation between Riemann-Liouville fractional integrals of type (α, k, P) and the corresponding operators of type $(\frac{\alpha}{k}, P)$:*

$$
\begin{aligned}
I_P^{(\alpha,k)} f(\mathbf{x}) &= \frac{1}{k\Gamma_k(\alpha)} \int_0^1 [s_P(\mathbf{x}) - s_{(P,\mathbf{x})}(t)]^{\frac{\alpha}{k}-1} f(\gamma_{\mathbf{x}}(t)) \big\|\gamma'_{\mathbf{x}}(t)\big\| dt\\
&= \frac{1}{k k^{\frac{\alpha}{k}-1}\Gamma(\alpha/k)} \int_0^1 [s_P(\mathbf{x}) - s_{(P,\mathbf{x})}(t)]^{\frac{\alpha}{k}-1} f(\gamma_{\mathbf{x}}(t)) \big\|\gamma'_{\mathbf{x}}(t)\big\| dt\\
&= \frac{1}{k^{\frac{\alpha}{k}}} \frac{1}{\Gamma(\alpha/k)} \int_0^1 [s_P(\mathbf{x}) - s_{(P,\mathbf{x})}(t)]^{\frac{\alpha}{k}-1} f(\gamma_{\mathbf{x}}(t)) \big\|\gamma'_{\mathbf{x}}(t)\big\| dt\\
&= k^{-\frac{\alpha}{k}} I_P^{\frac{\alpha}{k}} f(\mathbf{x}).
\end{aligned}
$$

This means that we can remit ourselves to Definition 4.1 to deduce results concerning $I_P^{(\alpha,k)}$.

Remark 6.2. *With respect to the function spaces associated to the (α, k, P) operators introduced above, they can be expressed in terms of the function spaces from Definitions 4.2 and 4.3, due to Remarks 6.1 and 6.3; for example, the space of functions whose Riemann-Liouville integral of type (α, k, P) exists on U is precisely $L^1_{(\alpha/k,P)}$. Interestingly, because of Corollary 4.1 we can affirm that for $\alpha, k > 0$ such that $0 < \alpha \le k$ we have $L^1_{(\alpha/k,P)}(U) \subseteq L^1_{(k/\alpha,P)}(U)$, i.e., the space of functions whose Riemann-Liouville integral of type (α, k, P) exists on U is contained in the space of functions whose Riemann-Liouville integral of type (k, α, P) exists on U, whenever $0 < \alpha \le k$.*

In fact, using Proposition 4.3, we obtain immediately the following.

Proposition 6.1. *Let $\alpha > 0$ be fixed and $P : U \to \mathfrak{C}^1(U)$ is any path choice. For $f \in L^1_{(\alpha/k,P)}(U)$, letting $k \to \infty$ we get $I_P^{(\alpha,k)} f(\mathbf{x}) \to f(\mathbf{x})$.*

Analogously to Definition 5.1,

Definition 6.2. *Take $\alpha, k > 0$, $n \in \mathbb{N}$ such that $n - 1 < \frac{\alpha}{k} \leq n$ and a path choice $P : U \to \mathfrak{C}^j(U)$. For $f : U \to \mathbb{R}$, the* ***Riemann-Liouville total differential of type*** *(α, k, P)* ***of*** *f* ***at*** *$\mathbf{x}$ is*

$$D_P^{(\alpha,k)} f(\mathbf{x})(\mathbf{v}_1, \ldots, \mathbf{v}_n) := k^n D^n I_P^{(nk-\alpha,k)} f(\mathbf{x})(\mathbf{v}_1, \ldots, \mathbf{v}_n). \tag{25}$$

Regarding Definition 6.2 with $\alpha \in (0, k]$, our definition of the total differential simplifies analogously:

$$D_P^{(\alpha,k)} f(\mathbf{x})(\mathbf{v}) := k \langle \nabla I_P^{(k-\alpha,k)} f(\mathbf{x}), \mathbf{v} \rangle. \tag{26}$$

Remark 6.3. *Following Remark 6.1, we have the following equivalence:*

$$\begin{aligned} D_P^{(\alpha,k)} f(\mathbf{x})(\mathbf{v}_1, \ldots, \mathbf{v}_n) &= k^n D^n I_P^{(nk-\alpha,k)} f(\mathbf{x})(\mathbf{v}_1, \ldots, \mathbf{v}_n) \\ &= k^n k^{-\frac{nk-\alpha}{k}} D^n I_P^{\frac{nk-\alpha}{k}} f(\mathbf{x})(\mathbf{v}_1, \ldots, \mathbf{v}_n) \\ &= k^{\frac{\alpha}{k}} D_P^{\frac{\alpha}{k}} f(\mathbf{x})(\mathbf{v}_1, \ldots, \mathbf{v}_n). \end{aligned}$$

Using the relationships between operators of type (α, k, P) and $(\alpha/k, P)$, it is immediate to extend the properties proven above. In particular, the fundamental theorems from the last section take the subsequent form.

Theorem 6.1. *Let $f : U \to \mathbb{R}$ and a path choice $P : U \to \mathfrak{C}^j(U)$ such that $I_P^{n-\alpha/k} f \in D_{(\alpha/k,P)}(U)$ with $n \in \mathbb{N}$ such that $n - 1 < \frac{\alpha}{k} \leq n$ with $\alpha > 0$. Then*

$$D_P^{(\alpha,k)} I_P^{(\alpha,k)} f(\mathbf{x}) \left(\mathbf{u}_\mathbf{x}\right)^n = f(\mathbf{x}). \tag{27}$$

Theorem 6.2. *Let $\alpha, k > 0$, $n \in \mathbb{N}$ with $n - 1 < \frac{\alpha}{k} \leq n$ and $P : U \to \mathfrak{C}^1(U)$. If $f : U \to \mathbb{R}$ satisfies $D_P^{(\alpha,k)} f(\cdot)(\mathbf{u}_{(\cdot)})^n \in C^n_{(\alpha/k,P)}(U)$, then the following identity holds:*

$$I_P^{(\alpha,k)} D_P^{(\alpha,k)} f(\mathbf{x})(\mathbf{u}_\mathbf{x})^n = f(\mathbf{x}) - \sum_{j=1}^{n} \frac{D_P^{(\alpha-kj,k)} f(\mathbf{a}_\mathbf{x})(\mathbf{w}_\mathbf{x})^{n-j}}{\Gamma_k(\alpha - kj + k)} s_P(\mathbf{x})^{\frac{\alpha}{k}-j}. \tag{28}$$

Proof. We use the relationships between the fractional operators of types (α, P) and (α, k, P), as well as equation (1):

$$\begin{aligned} I_P^{(\alpha,k)} D_P^{(\alpha,k)} f(\mathbf{x})(\mathbf{u}_\mathbf{x})^n &= I_P^{\alpha/k} D_P^{\alpha/k} f(\mathbf{x})(\mathbf{u}_\mathbf{x})^n \\ &= f(\mathbf{x}) - \sum_{j=1}^{n} \frac{D_P^{\alpha/k-j} f(\mathbf{a}_\mathbf{x})(\mathbf{w}_\mathbf{x})^{n-j}}{\Gamma(\alpha/k - j + 1)} s_P(\mathbf{x})^{\alpha/k-j} \\ &= f(\mathbf{x}) - \sum_{j=1}^{n} \frac{k^{\frac{\alpha-kj+k}{k}-1} D_P^{\alpha/k-j} f(\mathbf{a}_\mathbf{x})(\mathbf{w}_\mathbf{x})^{n-j}}{\Gamma_k(\alpha - kj + k)} s_P(\mathbf{x})^{\alpha/k-j} \\ &= f(\mathbf{x}) - \sum_{j=1}^{n} \frac{D_P^{(\alpha-kj,k)} f(\mathbf{a}_\mathbf{x})(\mathbf{w}_\mathbf{x})^{n-j}}{\Gamma_k(\alpha - kj + k)} s_P(\mathbf{x})^{\alpha/k-j}. \end{aligned}$$

□

Remark 6.4. *For $k = 1$, Theorems 6.1 and 6.2 reduce to Theorems 5.1 and 5.2, respectively.*

7 Conclusions

We reassessed the fractional trajectory integrals and differentials proposed in [27] via the Path Choice concept, which gave rise to the Riemann-Liouville trajectory operators of type (α, P). They recovered some partial fractional integrals and differentials from the classic theory and featured remarkable properties, analogous to those of their one-variable counterparts. In addition, we have considered an additional modification of the previous proposal via the additional perturbation parameter k, in order to produce (α, k, P) trajectory operators; this way, from the fundamental relations between the (α, P) and (α, k, P) operators, we were capable of extending the results for the type (α, P) to those concerning the type (α, k, P).

As expected, many other results and characteristics of the (α, k, P) operators remain to be explored in future efforts, as well as potential models to be implemented using our paradigm. Undoubtedly, the knowledge explored and exploited in this contribution enriches the area of Fractional Multivariable Calculus.

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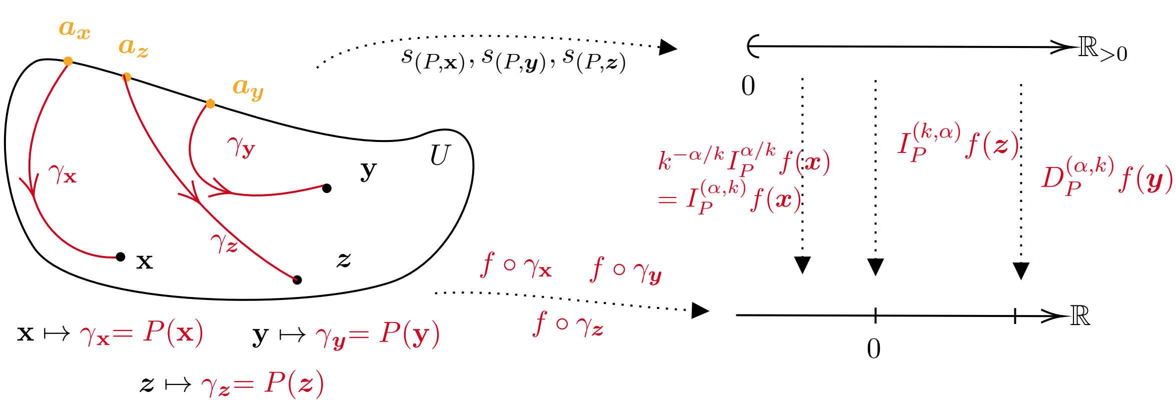
9 Conflict of Interest

The authors declare no conflicts of interest.

10 Artificial Intelligence Usage Statement

The authors declare that they have not used any generative artificial intelligence applications, software or websites for the writing of the present paper, the design of tables and figures or the analysis and interpretation of the data.

11 Graphical Abstract



12 Contribution Roles

Contribution	Author(s)
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