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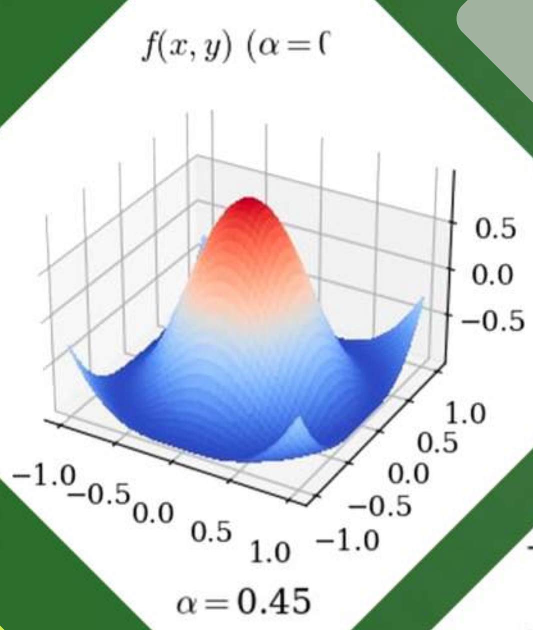

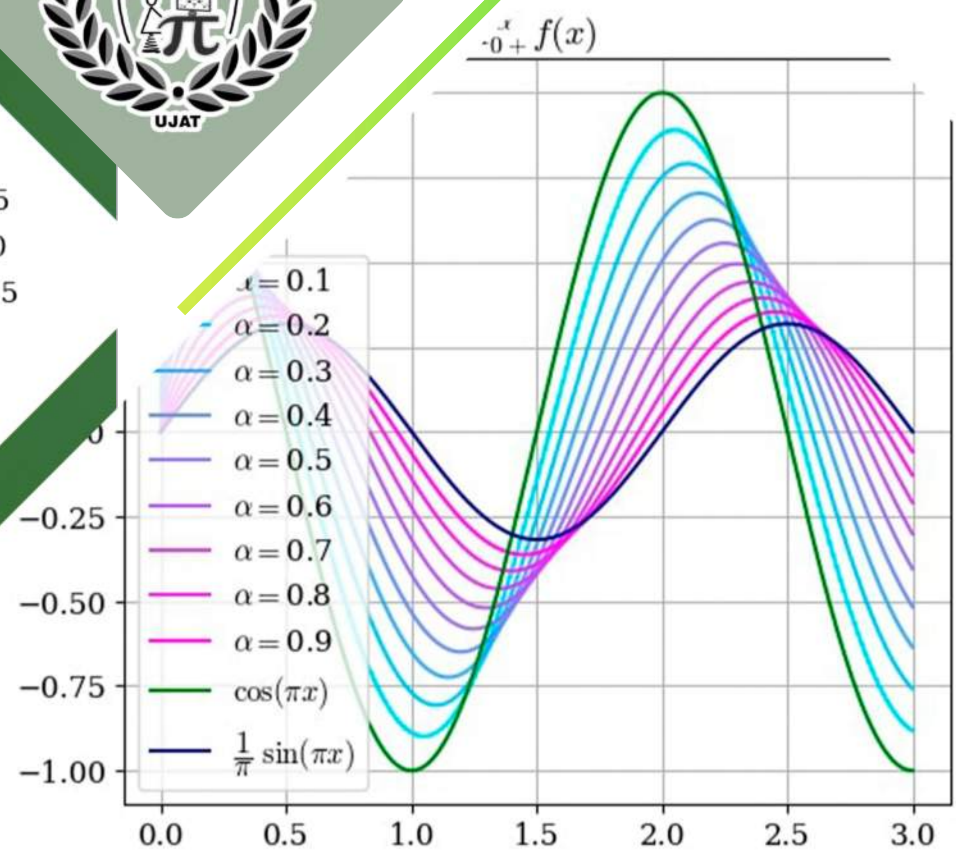

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Con el número 30 del Journal of Basic Sciences, se inicia el volumen 11 de esta revista correspondiente al año 2025. El carácter multidisciplinario de esta revista, permite enriquecer su contenido con perspectivas variadas que abordan diversas problemáticas en el área de las ciencias básicas y disciplinas afines.

De esta forma, se presenta una contribución que desarrolló generalizaciones en cálculo multivariable para llegar a nuevas diferenciales totales fraccionarias, las cuales juegan un papel importante en la modelación de gran número de fenómenos. Por otro lado, se incluye también una aportación que trata sobre el desarrollo de un método para resolver la ecuación de transporte conservativa en dominios específicos, incluyendo su validación y prueba para demostrar sus capacidades.

Se incluye además, un reporte encaminado a mejorar la calidad de imágenes, mediante técnicas de discretización numérica presentando una evaluación cualitativa y cuantitativa de los resultados obtenidos. En otro orden de ideas, se centra la atención hacia el estudio de sistemas aleatorios y la complejidad en su modelación, mostrando un estudio inferencial para un proceso de Poisson mixto, que lleva a la obtención de expresiones para densidad predictiva.

Es innegable que el aprendizaje de las matemáticas representa un reto actual que no debe soslayarse. En este sentido, se incluye un estudio que muestra la relación entre el desarrollo de la memoria de trabajo y el aprendizaje de identidades trigonométricas por parte de jóvenes del nivel medio superior, mostrando los subcomponentes necesarios en el razonamiento para el aprendizaje de este tema. En otra contribución relativa a la matemática educativa, se presenta una propuesta para atender el aprendizaje de los polígonos por estudiantes de bachillerato, mediante una serie de actividades diseñadas ex profeso que permiten una mejora en la comprensión de la temática.

En un contexto diferente, está el estudio dirigido a evaluar la actividad antibacteriana de extractos de plantas del género Cecropia, de uso tradicional en el sureste mexicano, correlacionando esta propiedad con el perfil fitoquímico analizado. Se presenta además, una contribución encaminada a analizar el impacto, que en los últimos años, han ocasionado derrames petroleros en el sureste mexicano, con especial énfasis en la afectación a cultivos agrícolas.

La atención de problemas de salud está dada a través de dos artículos que forman parte de este número. Por un lado, se comparó la resistencia a la insulina a través de índices específicos en momentos anteriores y durante la pandemia de COVID-19; en otro aporte, se analiza la relación entre diversos factores de riesgo asociados a la población joven y la enfermedad de Chagas. Mientras que en el área de la ciencia de los materiales, se incluye una propuesta para la obtención de derivados de poliuretano, con un método eficiente y compacto.

De esta forma el Journal of Basic Sciences acerca a sus lectores al amplio panorama del quehacer científico.

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On Riemann-Liouville Operators for Functions of One and Several Variables

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Resumen
En esta contribución, presentamos generalizaciones de las integrales de trayectorias del Cálculo Multivariable para desarrollar nuevas diferenciales totales fraccionarias, extendiendo así las diferenciales clásicas para funciones definidas en espacios Euclidianos. Exploramos y discutimos varios ejemplos y propiedades básicas para los nuevos operadores. Además, concluimos con un análisis comparativo de los operadores propuestos de cara a los ya existentes en la literatura.

Keywords: *Integral fraccional, derivada fraccional, operadores, trayectoria, diferencial de camino, parcial, generalización.*

Abstract
In this contribution, we present generalizations of path integrals from Multivariable Calculus to develop new fractional total differentials, extending the classical differentials for functions defined on Euclidean spaces. We explore and discuss several examples and key properties of these novel operators. Additionally, we conclude with a comparative analysis of our proposed operators alongside existing ones found in the literature.

Keywords: *Fractional Integral, Fractional Derivative, Operators, Trajectory, Path, Differential, Partial, Generalization.*

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1 Introduction

Fractional Calculus involves the analysis and application of differential and integral operators of arbitrary real or complex orders. Its origins trace back to mathematicians such as L'Hôpital and Leibniz. Since then, the field has experienced gradual yet steady growth (for a historical overview, see the introduction of [1], and additional data in Chapter 1 of [2]). Nowadays, Fractional Calculus plays a significant role in modeling a wide range of phenomena, including population dynamics [3, 4], electrical circuits [5, 6], epidemics [7, 8], and classical mechanics [9, 10], among others. The memory and non-locality properties of certain fractional operators [11], along with their potential to enhance the precision of models that traditionally rely on ordinary derivatives (e.g., [12, 13]), have driven the rapid expansion of both their study and application.

On the theoretical front, while substantial work has been done to demonstrate the potential of fractional operators across various fields (e.g., [1, 14, 15, 16]), much remains to be explored. For instance, a comprehensive and unified theory of Fractional Differential Geometry has yet to be developed. Although several contributions have been made in this area (e.g., [17, 18, 19, 20, 21, 22]), we agree that the development of such a theory may require the identification of more suitable fractional operators tailored to Euclidean spaces and, by extension, to differentiable manifolds.

The literature presents a vast array of ways to define fractional differential operators, and over the past 30 years, the diversity of such operators has expanded almost daily (see [23] for a compilation of various fractional operators). In this paper, we focus exclusively on **Riemann-Liouville-type fractional operators**, which were first introduced in J. Liouville's work [24] and Riemann's posthumous contributions [25]. Since then, these operators have become a foundational paradigm for the development of many other fractional derivatives.

In this study, we begin by reviewing the fundamental definitions and properties of Riemann-Liouville fractional integrals and derivatives. Building upon the classical concept of trajectory integrals for scalar functions, we introduce and investigate new Riemann-Liouville fractional operators that generalize both the trajectory integral and the Riemann-Liouville operators for single-variable functions. Lastly, after analyzing some key properties, we recover fractional operators for multivalued functions that have been previously proposed in the literature.

2 Riemann-Liouville Fractional Calculus

2.1 Fractional Calculus of One Variable

The definition of Riemann-Liouville fractional derivatives requires a preliminar notion of fractional integration.

Definition 2.1 ([1], p. 33)**.** *Let $f \in L^1([a,b])$ and $\alpha > 0$. The* ***Riemann-Liouville left fractional integral of f of order α at*** *$x \in [a,b]$ is defined as the function*

$$I_{a+}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt. \tag{1}$$

Similarly, the ***Riemann-Liouville right fractional integral of f of order α at*** *$x \in [a,b]$ is given as*

$$I_{b-}^{\alpha} f(x) := \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt. \tag{2}$$

A relationship between left and right fractional integrals can be obtained. For such purpose, let us define the *reflection operator* $Q : L^1([a,b]) \to L^1([a,b])$ as

$$Q(f(x)) = f(a+b-x).$$

Proposition 2.1. *For $f \in L^1([a,b])$, it is true that*

$$I_{b-}^{\alpha} f(x) = Q I_{a+}^{\alpha} Q f(x). \tag{3}$$

Proof. First, we see that

$$I_{a+}^{\alpha} Q f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} Q f(t)dt = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(a+b-t)dt.$$

If we let $u = a + b - t$, this change of variable transforms the integral into

$$\begin{aligned}\frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1} f(a+b-t)dt &= \frac{1}{\Gamma(\alpha)}\int_{a+b-x}^b (x-a-b+u)^{\alpha-1} f(u)du \\ &= I_{b-}^{\alpha} f(a+b-x) \\ &= QI_{b-}^{\alpha} f(x)\end{aligned}$$

and because $Q^2 f(x) = f(x)$, it follows that $I_{b-}^{\alpha} f(x) = QI_{a+}^{\alpha} Qf(x)$. □

Remark 2.1. *In [1], Theorem 2.7, it is proved that for $f \in L^1([a,b])$*

$$\lim_{\alpha \to 0^+} I_{a+}^{\alpha} f(x) = f(x)$$

almost everywhere (a.e.) in $[a,b]$ and considering Proposition 2.1,

$$\lim_{\alpha \to 0^+} I_{b-}^{\alpha} f(x) = f(x)$$

a.e. on $[a,b]$ too. For continuous functions on $[a,b]$, the foregoing limits hold everywhere on $[a,b]$ ([26], p. 66). In addition, it is known ([1], Theorem 2.6) that the (left and right) fractional integral of order α of $f \in L^1([a,b])$ (or $C([a,b])$) remains in $L^1([a,b])$ (resp. $C([a,b])$).

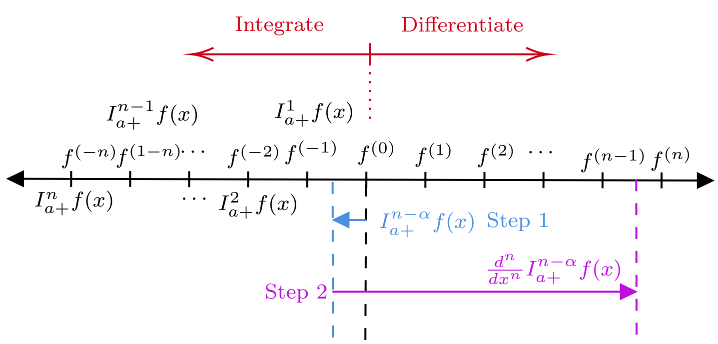

Figure 1: Graphical meaning behind Riemann-Liouville fractional derivatives' construction; forward steps correspond to differentiation and backward steps represent integration.

From Riemann-Liouville integration it is possible to construct a related fractional derivative: the intuitive idea is illustrated in Figure 1 and the corresponding definitions are given below. For the rest of the section, for any $\alpha > 0$, $n \in \mathbb{N}$ is such that $n - 1 < \alpha \le n$.

Definition 2.2 ([1], p. 37)**.** *Consider $\alpha > 0$, $\alpha \notin \mathbb{N}$ and $f \in C^n([a,b])$. Define the* ***Riemann-Liouville left fractional derivative of** f **of order** α **at** x as*

$$D_{a+}^{\alpha} f(x) := \frac{d^n}{dx^n} I_{a+}^{n-\alpha} f(x). \tag{4}$$

and ***Riemann-Liouville right fractional derivative of** f **of order** α **at** x is*

$$D_{b-}^{\alpha} f(x) := (-1)^n \frac{d^n}{dx^n} I_{b-}^{n-\alpha} f(x). \tag{5}$$

As expected, an analogous result to Proposition 2.1 holds between the left and right pair of Riemann-Liouville differential operators.

Proposition 2.2. *The following relationships between left and right Riemann-Liouville fractional derivatives holds:*

$$D_{b-}^{\alpha} f(x) = Q D_{a+}^{\alpha} Q f(x) \tag{6}$$

Proof. It is easy to see that $(Qf)^{(n)}(x) = (-1)^n Q f^{(n)}(x)$. Hence, by definition and Proposition 2.1,

$$Q D_{a+}^{\alpha} Q f(x) = Q(I_{a+}^{n-\alpha} Q f(x))^{(n)} = (-1)^n \left(Q I_{a+}^{n-\alpha} Q f(x)\right)^{(n)} = (-1)^n (I_{b-}^{n-\alpha} f(x))^{(n)}.$$

□

A main property that distinguishes Riemann-Liouville integration from Riemann-Liouville differentiation ([27], Property 2.4) is the classical **semigroup property**, whose proof can be found in [28], Theorem 2.2.

Theorem 2.1. *Let $f \in L^1([a,b])$ and $\alpha, \beta > 0$. Then it follows that*

$$\begin{aligned} I_{a+}^{\alpha} I_{a+}^{\beta} f(x) &= I_{a+}^{\alpha+\beta} f(x) \quad \text{and} \\ I_{a+}^{\alpha} I_{a+}^{\beta} f(x) &= I_{a+}^{\alpha+\beta} f(x) \end{aligned} \tag{7}$$

a.e. on $[a,b]$. Furthermore, if $\alpha + \beta \geq 1$ or $f \in C([a,b])$, the identities (7) hold everywhere on $[a,b]$.

Remark 2.2. *From Remark 2.1 it follows that*

$$\lim_{\alpha \to n^-} D_{a+}^{\alpha} f(x) = D^n f(x) \tag{8}$$

and

$$\lim_{\alpha \to n^-} D_{a+}^{\alpha} f(x) = (-1)^n D^n f(x). \tag{9}$$

As a consequence, it is custom to define $D_{a+}^{n} f(x) := D^n f(x)$ and $D_{b-}^{n} f(x) := (-1)^n D^n f(x)$.

The Riemann-Liouville fractional operators are clearly linear and reduce to the classical differential and integral operators when the order is an integer. However, these operators possess additional noteworthy properties. Among the many interesting characteristics they exhibit, we highlight a key result, proved in Lemmas 2.5, 2.21a, and 2.22 of [27], which extends the classical Fundamental Theorem of Calculus to the realm of Riemann-Liouville fractional derivatives.

Theorem 2.2. *Let $f : [a,b] \to \mathbb{R}$ and $\alpha > 0$.*

1. *If $f \in C([a,b])$, then*

$$D_{a+}^{\alpha} I_{a+}^{\alpha} f(x) = f(x) \quad \text{and} \quad D_{b-}^{\alpha} I_{b-}^{\alpha} f(x) = f(x). \tag{10}$$

2. *For $f \in C^n([a,b])$, the following identities hold for any $x \in [a,b]$:*

$$I_{a+}^{\alpha} D_{a+}^{\alpha} f(x) = f(x) - \sum_{k=0}^{n-1} \frac{D_{a+}^{\alpha-k} f(a)}{\Gamma(\alpha-k+1)} (x-a)^{\alpha-k}, \tag{11}$$

$$I_{b-}^{\alpha} D_{b-}^{\alpha} f(x) = f(x) - \sum_{k=0}^{n-1} \frac{(-1)^k D_{b-}^{\alpha-k} f(b)}{\Gamma(\alpha-k+1)} (b-x)^{\alpha-k}. \tag{12}$$

Example 2.1. *Let us consider $f(x) = \cos(\omega x)$ with $\omega \neq 0$. Using Theorems 2.7 and 2.15 from [28] with the Taylor series of $f(x)$ around zero, it can be proved that*

$$I_{0+}^{\alpha} f(x) = x^{\alpha} E_{2,1+\alpha}(-\omega^2 x^2) \text{ and } D_{0+}^{\beta} f(x) = x^{-\beta} E_{2,1-\beta}(-\omega^2 x^2) \tag{13}$$

for $x > 0$, $\alpha > 0$ and $0 < \beta < 1$, where the expression $E_{r,s}(x)$ stands for the so-called ***two-parameter Mittag-Leffler function***

$$E_{r,s}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(rn+s)}.$$

It is known (Theorem 4.1 from [28]) that a sufficient condition for the uniform convergence of such function for any $x > 0$ is $r, s > 0$ (this is a reason why we considered $\beta \in (0,1)$). Some particular cases are shown in Figure 2.

2.2 Fractional Calculus of Several Variables

Similar to the univariate case, there is no single, widely accepted convention in the literature regarding the definition of fractional integrals and derivatives for real functions defined on subsets of $\mathbb{R}^m$ (for example, see [1], Chapter 5, which discusses several different approaches). To address this, and to properly recover the classical integer-order differentials from Multivariable Calculus, we introduce new multivariable fractional operators inspired by those proposed in [29].

From [30, p. 401] we remember that the path integral of $f : U \subseteq \mathbb{R}^m \to \mathbb{R}$ over a continuously differentiable curve $\gamma : [t_0, t_1] \to \mathbb{R}$ (also known as the line integral of f with respect to the arc length of γ) is defined as

$$\int_{\gamma} f \; ds = \int_{t_0}^{t_1} f(\gamma(t)) \|\gamma'(t)\| dt.$$

Notice that when we reduce the dimension of the ambient space to $m = 1$, we recover the classical integral over some interval $[a, b]$, where the line segment connecting the values a and b plays the the role of the curve γ. We would like to work with an analogous fractional integral, depending on the path γ. In this sense, let

$$I^1(f \circ \gamma)(t^*) := \int_{t_0}^{t^*} f(\gamma(t)) \|\gamma'(t)\| dt$$

for some $t^* \in [t_0, t_1]$ and let $I^n(f \circ \gamma)(t^*)$ for $n \in \mathbb{N}$, $n \geq 2$ be defined recursively as $I^n(f \circ \gamma)(t^*) := I^{n-1}\left(I^1(f \circ \gamma)\right)(t^*)$.

Theorem 2.3. *For $f \in C(U)$ with $U \subseteq \mathbb{R}^m$ open and path-connected, $\gamma \in C^1([t_0, t_1], U)$ and $t^* \in [t_0, t_1]$, the following Cauchy-type iterated integration formula holds:*

$$I^n(f \circ \gamma)(t^*) = \frac{1}{(n-1)!} \int_{t_0}^{t^*} [s(t^*) - s(t)]^{n-1} f(\gamma(t)) \|\gamma'(t)\| dt, \tag{14}$$

where $s(t) = \int_{t_0}^{t} \|\gamma'(t)\| dt$ stands for the arc length of γ at t.

Proof. It follows by induction: it is true for $n = 1$ by definition and we suppose that it is true for $n - 1$. Thus,

$$
\begin{aligned}
I^n(f \circ \gamma)(t^*) &= I^{n-1}(I^1(f \circ \gamma))(t^*) \\
&= \frac{1}{(n-2)!} \int_{t_0}^{t^*} [s(t^*) - s(t)]^{n-2}\, I^1(f \circ \gamma)(t) \|\gamma'(t)\| dt \\
&= \frac{1}{(n-2)!} \int_{t_0}^{t^*} \int_{t_0}^{t} [s(t^*) - s(t)]^{n-2}\, f \circ \gamma(\tau) \|\gamma'(\tau)\| \|\gamma'(t)\| d\tau \; dt \\
&= \frac{1}{(n-2)!} \int_{t_0}^{t^*} \int_{\tau}^{t^*} [s(t^*) - s(t)]^{n-2}\, f \circ \gamma(\tau) \|\gamma'(\tau)\| \|\gamma'(t)\| dt \; d\tau \\
&= \frac{1}{(n-2)!} \int_{t_0}^{t^*} f \circ \gamma(\tau) \|\gamma'(\tau)\| \left(\int_{\tau}^{t^*} [s(t^*) - s(t)]^{n-2} \|\gamma'(t)\| dt \right) d\tau \\
&= \frac{1}{(n-2)!} \int_{t_0}^{t^*} f \circ \gamma(\tau) \|\gamma'(\tau)\| \left(\int_{t^*}^{\tau} \frac{\mathrm{d}}{\mathrm{dt}} \left(\frac{[s(t^*) - s(t)]^{n-1}}{n-1} \right) dt \right) d\tau \\
&= \frac{1}{(n-1)!} \int_{t_0}^{t^*} [s(t^*) - s(\tau)]^{n-1}\, f \circ \gamma(\tau) \|\gamma'(\tau)\| d\tau
\end{aligned}
$$

and the theorem follows. □

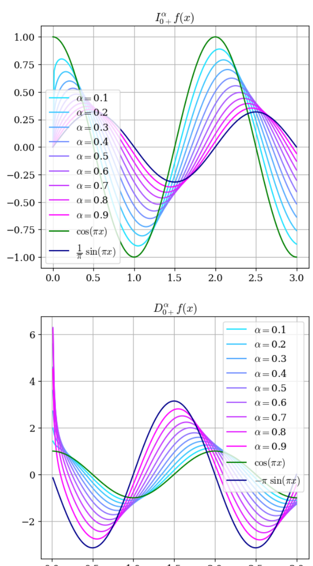

Figure 2: Some left Riemann-Liouville integrals and derivatives for Example 2.1.

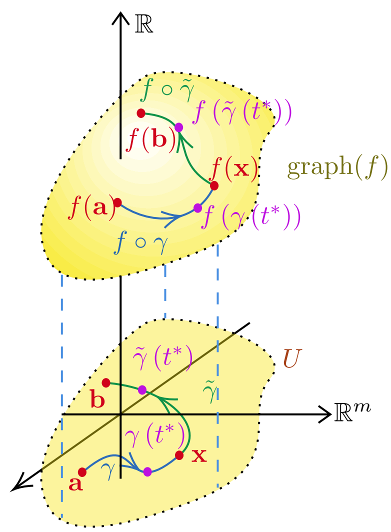

Figure 3: Geometrical motivation for a definition of Riemann-Liouville fractional integration on $\mathbb{R}^m$.

For the rest of the section, let $U \subseteq \mathbb{R}^m$ be some open and path-connected set and let $\mathbf{a}, \mathbf{b} \in U$ be fixed points. In addition, for each $\mathbf{x} \in U$ let $\gamma, \tilde{\gamma} : [t_0, t_1] \to U$ be regular ($\gamma'(t) \neq \mathbf{0}$ and $\tilde{\gamma}'(t) \neq \mathbf{0}$) and continuously differentiable curves with $\gamma(t_0) = \mathbf{a}$, $\gamma(t_1) = \tilde{\gamma}(t_0) = \mathbf{x}$ and $\tilde{\gamma}(t_1) = \mathbf{b}$ (Figure 3). Also, when it may be convenient, we will drop the tilde from the curve $\tilde{\gamma}$, in order to distinguish future definitions and results by the signs $+$ and $-$. In this way, based on equation (14), we propose the following definition.

Definition 2.3. *Let $\alpha > 0$ and $f \in C(U)$. For $t^* \in [t_0, t_1]$, the* ***Riemann-Liouville left fractional integral of*** *f* ***of order*** *α* ***at*** *$\gamma(t^*)$* ***along*** *γ is defined as*

$$I_+^\alpha(f \circ \gamma)(t^*) := \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t^*} [s(t^*) - s(t)]^{\alpha-1} f(\gamma(t)) \left\| \gamma'(t) \right\| dt \tag{15}$$

and the ***Riemann-Liouville right fractional integral of*** *f* ***of order*** *α* ***at*** *$\tilde{\gamma}(t^*)$* ***along*** *$\tilde{\gamma}$ is given by*

$$I_-^\alpha(f \circ \tilde{\gamma})(t^*) := \frac{1}{\Gamma(\alpha)} \int_{t^*}^{t_1} [\tilde{s}(t) - \tilde{s}(t^*)]^{\alpha-1} f(\tilde{\gamma}(t)) \left\| \tilde{\gamma}'(t) \right\| dt. \tag{16}$$

Here, $s(t) = \int_{t_0}^t \|\gamma'(\tau)\| d\tau$ and $\tilde{s}(t) = \int_{t_0}^t \|\tilde{\gamma}'(\tau)\| d\tau$ are the arc-lengths of γ and $\tilde{\gamma}$ at the parameter t, respectively. In particular, we denote

$$I_{\gamma+}^\alpha f(\mathbf{x}) := I_+^\alpha (f \circ \gamma)(t_1) \tag{17}$$

and

$$I_{\tilde{\gamma}-}^\alpha f(\mathbf{x}) := I_-^\alpha (f \circ \tilde{\gamma})(t_0). \tag{18}$$

Proposition 2.3. *Under the hypotheses of Definition 2.3, we can express (17,18) as*

$$I_{\gamma+}^\alpha f(\mathbf{x}) = [s(t_1)]^\alpha I_{0+}^\alpha \varphi(1) \tag{19}$$

and

$$I_{\tilde{\gamma}-}^\alpha f(\mathbf{x}) = [\tilde{s}(t_1)]^\alpha I_{1-}^\alpha \tilde{\varphi}(0), \tag{20}$$

respectively, where $\varphi, \tilde{\varphi} \in C([0,1])$ are defined as

$$\varphi(t) := f\left[\gamma\left(s^{-1}\left(s(t_1)t\right)\right)\right] \quad \text{and} \quad \tilde{\varphi}(t) := f\left[\tilde{\gamma}\left(\tilde{s}^{-1}\left(\tilde{s}(t_1)t\right)\right)\right],$$

respectively.

Proof. Because γ and $\tilde{\gamma}$ are assumed to be regular on $[t_0, t_1]$, both $\frac{ds}{dt}$ and $\frac{d\tilde{s}}{dt}$ are nonzero, whence s and $\tilde{s}$ are invertible on (t_0, t_1) by the classical inverse function theorem (e.g. p. 197 from [31]). Hence, we can apply the respective variable changes $w = \frac{s(t)}{s(t^*)}$ and $w = \frac{\tilde{s}(t) - \tilde{s}(t^*)}{\tilde{s}(t_1) - \tilde{s}(t^*)}$ to the integrals (15,16) and it is true that

$$I_+^\alpha(f \circ \gamma)(t^*) = s(t^*)^\alpha I_{0+}^\alpha \varphi(1) \tag{21}$$

and

$$I_-^\alpha(f \circ \tilde{\gamma})(t^*) = [\tilde{s}(t_1) - \tilde{s}(t^*)]^\alpha I_{1-}^\alpha \tilde{\varphi}(0), \tag{22}$$

with $\varphi(t) = f\left[\gamma\left(s^{-1}\left(s(t^*)t\right)\right)\right]$ and $\tilde{\varphi}(t) = f\left[\tilde{\gamma}\left(\tilde{s}^{-1}\left((1-t)\tilde{s}(t^*) + \tilde{s}(t_1)t\right)\right)\right]$. Finally, by considering $t^* = t_1$ and $t^* = t_0$, respectively, we get (19, 20). $\square$

The extension proposed in Definition 2.3 also satisfies a semigroup-like property, which simultaneously extents Theorem 2.1.

Theorem 2.4. *Set $f \in C(U)$. Under the assumptions of Definition 2.3, it is true that*

$$I_{\gamma\pm}^\alpha I_{\gamma\pm}^\beta f(\mathbf{x}) = I_{\gamma\pm}^{\alpha+\beta} f(\mathbf{x}). \tag{23}$$

Proof. In order to prove this result, we follow the same approach as the one used for Theorem 2.2 from [28]. Let us start with the left case. By a direct computation and the use of Fubini's theorem (c.f. [32, Corollary 7]), we have that

$$\begin{aligned}
I^{\alpha}_{\gamma+}(I^{\beta}_{\gamma+}f\circ\gamma)(t^*) &= \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t^*}[s(t^*)-s(t)]^{\alpha-1}\, I^{\beta}_{\gamma+}f(\gamma(t))\|\gamma'(t)\|dt \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{t_0}^{t^*}\int_{t_0}^{t}[s(t^*)-s(t)]^{\alpha-1}[s(t)-s(\tau)]^{\beta-1} f(\gamma(\tau))\|\gamma'(\tau)\|\|\gamma'(t)\|d\tau\, dt \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{t_0}^{t^*}\int_{\tau}^{t^*}[s(t^*)-s(t)]^{\alpha-1}[s(t)-s(\tau)]^{\beta-1} f(\gamma(\tau))\|\gamma'(\tau)\|\|\gamma'(t)\|dt\, d\tau \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_{t_0}^{t^*} f(\gamma(\tau))\|\gamma'(\tau)\|\left(\int_{\tau}^{t^*}[s(t^*)-s(t)]^{\alpha-1}[s(t)-s(\tau)]^{\beta-1}\|\gamma'(t)\|dt\right)d\tau.
\end{aligned}$$

The variable change $w := \frac{s(t^*)-s(t)}{s(t^*)-s(\tau)}$ turns the inner integral of the last line into

$$[s(t^*)-s(\tau)]^{\alpha+\beta-1}\,\mathrm{B}(\alpha,\beta),$$

where $\mathrm{B}(\alpha,\beta)$ is the beta function

$$\mathrm{B}(a,b) = \int_0^1 w^{a-1}(1-w)^{b-1}dw$$

and by the multiplicative identity

$$\mathrm{B}(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)},$$

(c.f. [33, p. 521]) we obtain $I^{\alpha+\beta}_{\gamma+}(f\circ\gamma)(t^*)$. Finally, letting $t^* = t_1$ we find (23) for the left case. The proof for the right identity can be carried out similarly, by considering the variable change $w := \frac{\tilde{s}(t)-\tilde{s}(t^*)}{\tilde{s}(\tau)-\tilde{s}(t^*)}$. □

The following result generalizes Remark 2.1.

Theorem 2.5. *Let $f \in C(U)$ and $\alpha > 0$. We have that*

$$\lim_{\alpha\to 0^+} I^{\alpha}_{\gamma\pm}f(\mathbf{x}) = f(\mathbf{x}). \tag{24}$$

Proof. It is a direct consequence of Remark 2.1 and Definition 2.3, because

$$\begin{aligned}
\lim_{\alpha\to 0^+} I^{\alpha}_{\gamma+}f(\mathbf{x}) &= \lim_{\alpha\to 0^+}[s(t_1)]^{\alpha} I^{\alpha}_{0+}\varphi(1) = 1\cdot\varphi(1) = f(\mathbf{x}), \\
\lim_{\alpha\to 0^+} I^{\alpha}_{\gamma-}f(\mathbf{x}) &= \lim_{\alpha\to 0^+}[\tilde{s}(t_1)]^{\alpha} I^{\alpha}_{1-}\tilde{\varphi}(0) = 1\cdot\tilde{\varphi}(0) = f(\mathbf{x}).
\end{aligned}$$

□

Remark 2.3. *Analogously to Remark 2.1, the previous result allows us to adopt the conventions $I^{0}_{\gamma\pm}f(\mathbf{x}) := f(\mathbf{x})$.*

Example 2.2. *Let us consider $f : \mathbb{R}^m \to \mathbb{R}$ defined as $f(\mathbf{x}) = \cos(\omega\|\mathbf{x}\|)$ and let $\gamma : [0,1] \to \mathbb{R}^n$ be the straight line $\gamma(t) = \|\mathbf{x}\|t$. Given that $s(t) = \|\mathbf{x}\|t$, $\|\gamma'(t)\| = \|\mathbf{x}\|$ and $f(\gamma(s^{-1}(s(1)t))) = f(t\mathbf{x})$, the computation of $I^{\alpha}_{\gamma+}f(\mathbf{x})$ for $\alpha > 0$ yields the following, using Proposition 2.3 and Example 2.1:*

$$I^{\alpha}_{\gamma+}f(\mathbf{x}) = \|\mathbf{x}\|^{\alpha}E_{2,1+\alpha}(-\omega^2\|\mathbf{x}\|^2). \tag{25}$$

Some plots are shown in Figure 4 for $m = 2$ and $\omega = \pi$.

From Multivariable Calculus it is known (e.g. [34, p. 192]) that for $f \in C^k(U)$ with $U \subseteq \mathbb{R}^m$ open, the total differential of f of order k at $\mathbf{x} \in U$ constitutes a symmetric[1] k-tensor $D^k f(\mathbf{x}) : \prod_{i=1}^k T_{\mathbf{x}} U \to \mathbb{R}$ with explicit expression[2]

$$D^k f(\mathbf{x})(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \sum_{i_1,\ldots,i_k=1}^{m} v_{1,i_1} \cdots v_{k,i_k} \frac{\partial^k f(\mathbf{x})}{\partial x_{i_1} \cdots \partial x_{i_k}}.$$

We can use this fact and the Riemann-Liouville multivariable integrals from Definition 2.3 to introduce a multivariable generalization of Riemann-Liouville fractional derivatives (4,5), which simultaneously generalizes the total differential of k-th order to real positive orders.

Definition 2.4. *Take $\alpha > 0$, $\alpha \notin \mathbb{N}$, $k \in \mathbb{N}$ such that $k-1 < \alpha < k$ and f such that $I_+^{k-\alpha} f \circ \gamma$ and $I_-^{k-\alpha} f \circ \gamma$ belong to $C^k(U)$. For $t^* \in [t_0, t_1]$, the **Riemann-Liouville left differential of order α of f at $\gamma(t^*)$ along γ** is*

$$D_+^\alpha (f \circ \gamma)(t^*)(\mathbf{v}_1, \ldots, \mathbf{v}_k) := D^k I_+^{k-\alpha} (f \circ \gamma)(t^*)(\mathbf{v}_1, \ldots, \mathbf{v}_k). \tag{26}$$

*and the **Riemann-Liouville right differential of f of order α at $\tilde{\gamma}(t^*)$ along $\tilde{\gamma}$** is given by*

$$D_-^\alpha (f \circ \tilde{\gamma})(t^*)(\mathbf{v}_1, \ldots, \mathbf{v}_k) := (-1)^k D^k I_-^{k-\alpha} (f \circ \tilde{\gamma})(t^*)(\mathbf{v}_1, \ldots, \mathbf{v}_k). \tag{27}$$

In particular, for $t^* = t_1$ and $t^* = t_0$, we respectively denote

$$D_{\gamma+}^\alpha f(\mathbf{x})(\mathbf{v}_1, \ldots, \mathbf{v}_k) := D_+^\alpha (f \circ \gamma)(t_1)(\mathbf{v}_1, \ldots, \mathbf{v}_k) \tag{28}$$

and

$$D_{\tilde{\gamma}-}^\alpha f(\mathbf{x})(\mathbf{v}_1, \ldots, \mathbf{v}_k) := D_-^\alpha (f \circ \tilde{\gamma})(t_0)(\mathbf{v}_1, \ldots, \mathbf{v}_k). \tag{29}$$

As a special case $\alpha \in (0, 1]$ (also known among the literature as the *regime of low orders*), our definition of the total differential reduces to the computation of a usual gradient:

$$D_{\gamma\pm}^\alpha f(\mathbf{x})(\mathbf{v}) := \langle \nabla I_{\gamma\pm}^{1-\alpha} f(\mathbf{x}), \mathbf{v} \rangle. \tag{30}$$

Remark 2.4. *The notions we proposed in Definition 2.4 could allow us to provide a nice approach that extends the classical Taylor expansion for (sufficiently) smooth functions on open sets around points in Euclidean spaces. In fact, from Theorem 2.5, it follows that*

$$\lim_{\alpha \to k^-} D_{\gamma+}^\alpha f(\mathbf{x})(\mathbf{v}_1, \ldots, \mathbf{v}_k) = D^k f(\mathbf{x})(\mathbf{v}_1, \ldots, \mathbf{v}_k) \tag{31}$$

and

$$\lim_{\alpha \to k^-} D_{\gamma-}^\alpha f(\mathbf{x})(\mathbf{v}_1, \ldots, \mathbf{v}_k) = (-1)^k D^k f(\mathbf{x})(\mathbf{v}_1, \ldots, \mathbf{v}_k). \tag{32}$$

Hence, for $n \in \mathbb{N}$ we define

$$D_{\gamma+}^n f(\mathbf{x})(\mathbf{v}_1, \ldots, \mathbf{v}_n) := D^n f(\mathbf{x})(\mathbf{v}_1, \ldots, \mathbf{v}_n)$$

and

$$D_{\gamma-}^n f(\mathbf{x})(\mathbf{v}_1, \ldots, \mathbf{v}_n) := (-1)^n D^n f(\mathbf{x})(\mathbf{v}_1, \ldots, \mathbf{v}_n).$$

[1] Due to the commutativity of mixed partial derivatives when $k \geq 2$.

[2] $T_{\mathbf{x}} U$ is the so-called **tangent space to U at $\mathbf{x}$** and geometrically, it can be thought as the product space $\{\mathbf{x}\} \times \mathbb{R}^m$.

As a generalization of equations (10) in Theorem 2.2, we have the following result for low orders.

Theorem 2.6. *For $f \in C(U)$ and $\alpha \in (0,1]$, if $\mathbf{v} := \gamma'(t_1)$, then*

$$D^{\alpha}_{\gamma\pm} I^{\alpha}_{\gamma\pm} f(\mathbf{x}) \left(\frac{\mathbf{v}}{\|\mathbf{v}\|} \right) = f(\mathbf{x}). \tag{33}$$

Proof. By definitions (26) and (27), Theorem 2.1 and the classical multivariable chain rule (e.g. [30, Theorem 8.8]),

$$D^{\alpha}_{\pm} I^{\alpha}_{\pm} (f \circ \gamma)(t^*) (\gamma'(t^*)) = \sum_{i=1}^{m} \frac{\partial \left(I^1_{\pm} (f \circ \gamma)(t^*) \right)}{\partial x_i} \gamma'_i(t^*) = \frac{d}{dt} \left(I^1_{\pm} (f \circ \gamma)(t) \right) \Bigg|_{t=t^*} = \left\| \gamma'(t^*) \right\| f(\gamma(t^*)).$$

If we set $t^* = t_1$ for the left case and $t^* = t_0$ in the right case, the theorem follows. □

We must mention that, due to Theorems 2.5, 2.1, 2.6 and Remark 2.4, the family of operators $\{D^{\alpha}_{\gamma\pm}, I^{\alpha}_{\gamma\pm}\}$ for $\alpha \in (0,1]$ satisfies the desiderata for fractional integrals and derivatives proposed by Hilfer and Luchko in [35], which supports the validity of Definitions 2.3 and 2.4.

A definition for Riemann-Liouville (left and right) partial derivatives results immediately from equations (30):

$$\frac{\partial^{\alpha}_{\gamma\pm} f(\mathbf{x})}{\partial x_{i_1} \cdots \partial x_{i_m}} := D^{\alpha}_{\gamma\pm} f(\mathbf{x})(\hat{\mathbf{e}}_{i_1}, \ldots, \hat{\mathbf{e}}_{i_m}) = (\pm 1)^k \frac{\partial^k (I^{k-\alpha}_{\gamma\pm} f(\mathbf{x}))}{\partial x_{i_1} \cdots \partial x_{i_m}} \tag{34}$$

As a consequence, by linearity of the ordinary partial derivatives and the fractional integrals from Definition 2.3, we can express the left and right differentials of order α explicitly as

$$D^{\alpha}_{\gamma\pm} f(\mathbf{x})(\mathbf{v}_1, \ldots, \mathbf{v}_n) = \sum_{i_1, \ldots, i_k = 1}^{m} v_{1,i_1} \cdots v_{k,i_k} \frac{\partial^{\alpha}_{\gamma\pm} f(\mathbf{x})}{\partial x_{i_1} \cdots \partial x_{i_n}} \tag{35}$$

In particular, for $0 < \alpha \leq 1$,

$$D^{\alpha}_{\gamma\pm} f(\mathbf{x})(\mathbf{v}) = \langle \nabla^{\alpha}_{\gamma\pm} f(\mathbf{x}), \mathbf{v} \rangle \tag{36}$$

where we have naturally defined the **left and right Riemann-Liouville gradient vectors of f of order α at $\mathbf{x}$ along γ** as

$$\nabla^{\alpha}_{\gamma\pm} f(\mathbf{x}) := \left(\frac{\partial^{\alpha}_{\gamma\pm} f(\mathbf{x})}{\partial x_1}, \cdots, \frac{\partial^{\alpha}_{\gamma\pm} f(\mathbf{x})}{\partial x_n} \right). \tag{37}$$

Definition (37) motivates a generalization of the classical Jacobian matrix for continuously differentiable functions $f : \mathbb{R}^m \to \mathbb{R}^p$.

Definition 2.5. *Let $\alpha_i \in (0,1]$ for $i = 1, \ldots, p$ and $f \in C^1(U, \mathbb{R}^p)$. The **left and right Riemann-Liouville Jacobian matrices of order $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_p)$** are defined as*

$$J^{\alpha}_{\gamma\pm} f(\mathbf{x}) := \left(\frac{\partial^{\alpha_i}_{\gamma\pm} f_i(\mathbf{x})}{\partial x_j} \right)^{p,m}_{i,j=1} \tag{38}$$

If we define the Riemann-Liouville fractional differential of $f \in C^1(U, \mathbb{R}^p)$ of order $\boldsymbol{\alpha}$ as the vector function

$$D^{\boldsymbol{\alpha}}_{\gamma\pm} f(\mathbf{x}) := \left(D^{\alpha_1}_{\gamma\pm} f_1(\mathbf{x}), \ldots, D^{\alpha_p}_{\gamma\pm} f_p(\mathbf{x}) \right), \tag{39}$$

then for $\alpha_i \in (0,1]$, the Riemann-Liouville Jacobian matrices (39) of order $\boldsymbol{\alpha}$ are the associated matrices of the linear transformations defined by $D^{\boldsymbol{\alpha}}_{\gamma\pm} f(\mathbf{x})(\mathbf{v})$ with respect to the canonical basis.

Example 2.3. *Let* $f(\mathbf{x}) = \cos(\omega\|\mathbf{x}\|)$, $\alpha \in (0,1)$ *and assume the rest of hypotheses from Example 2.2. Then it follows that*

$$I^{1-\alpha}_{\gamma+} f(\mathbf{x}) = \|\mathbf{x}\|^{1-\alpha} E_{2,2-\alpha}(-\omega^2\|\mathbf{x}\|^2).$$

By Theorem 2.15 from [28], we can differentiate the foregoing series termwise. For such purpose, it is important to know that for any $\beta \in \mathbb{R}$ *and* $i = 1, \dots, m$,

$$\frac{\partial}{\partial x_i}\|\mathbf{x}\|^{\beta} = \beta\|\mathbf{x}\|^{\beta-2} x_i.$$

Hence, it results that for each $i = 1, \dots, n$ *and* $\mathbf{x} \neq \mathbf{0}$,

$$\frac{\partial^{\alpha}_{\gamma+} f(\mathbf{x})}{\partial x_i} = \frac{x_i}{\|\mathbf{x}\|^{1+\alpha}} E_{2,1-\alpha}(-\omega^2\|\mathbf{x}\|^2). \tag{40}$$

Therefore,

$$\nabla^{\alpha}_{\gamma+} f(\mathbf{x}) = \frac{1}{\|\mathbf{x}\|^{1+\alpha}} E_{2,1-\alpha}(-\omega^2\|\mathbf{x}\|^2)\mathbf{x}. \tag{41}$$

Some plots of the Riemann-Liouville partial derivatives of f *along* γ *are shown in Figure 5, for* $m = 2$ *and* $\omega = \pi$ *as in Example 2.2.*

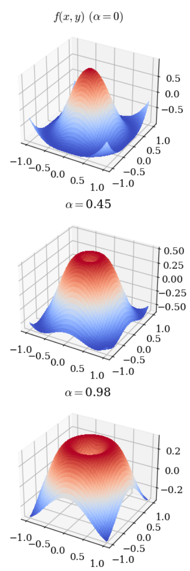

Figure 4: Example for the Riemann-Liouville path integral of some orders α for the function of Example 2.2.

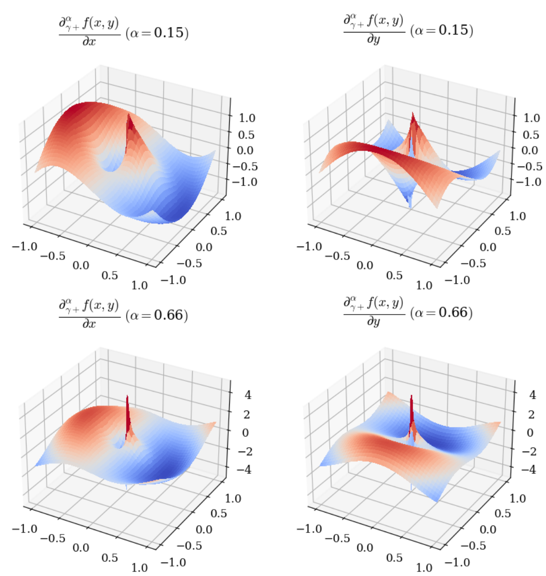

Figure 5: Graphs of some fractional partial derivatives of the function chosen in Example 2.3.

Remark 2.5. *To conclude this section, we point out that our definitions recover some already proposed throughout the literature: for example, the Cheng-Dai fractional integrals proposed in [29] result as a particular case of ours; such operators are defined over straight lines analogously to our Examples 2.2 and 2.3: specifically, let* $\gamma : [0,1] \to U$ *be defined as* $\gamma(t) = \mathbf{a} + (\mathbf{x} - \mathbf{a})t$ *for some* $\mathbf{a}, \mathbf{x}$ *in the convex domain* $U \subseteq \mathbb{R}^m$ *of some* $f \in C(U)$. *Then the* ***Cheng-Dai fractional integral of*** f ***at*** $\mathbf{x}$ ***of order*** $\alpha > 0$ ***along the directed segment*** γ *is given as*

$$\mathcal{I}^{\alpha}_{\mathbf{a}} f(\mathbf{x}) = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1} f(\gamma(t))dt. \tag{42}$$

It is easy to see that for such choice of γ *for each* $\mathbf{x}$

$$\mathcal{I}^{\alpha}_{\mathbf{a}} f(\mathbf{x}) = \|\mathbf{x} - \mathbf{a}\|^{-\alpha} I^{\alpha}_{\gamma+} f(\mathbf{x}), \tag{43}$$

which can be verified using Proposition 2.3. On the other hand, more commonly, an accepted definition for the Riemann-Liouville partial fractional operators can be found in various sources, such as section 2.9 from [27], [14, p. 251], section 24.2 from [1] or the article [17]:

$$\mathbb{I}^{\alpha}_{a_i+} f(\mathbf{x}) = \frac{1}{\Gamma(\alpha)} \int_{a_i}^{x_i} \frac{f(x_1, \ldots, \overbrace{t}^{i}, \ldots, x_m)}{(x_i - t)^{1-\alpha}} dt, \tag{44}$$

whence a Riemann-Liouville partial fractional derivative of order $0 < \alpha < 1$ *can be constructed:*

$$\mathbb{D}^{\alpha}_{a_i+} f(\mathbf{x}) = \frac{\partial \mathbb{I}^{1-\alpha}_{a_i+} f(\mathbf{x})}{\partial x_i}. \tag{45}$$

For the previous notions to be defined, it is custom to have $U \supseteq \prod_{i=1}^{m} [a_i, x_i]$. *Now it is easy to see that if* $\gamma : [0,1] \to \mathbb{R}^n$ *is the line segment* $\gamma(t) = (a_i + (x_i - a_i)t)\hat{\mathbf{e}}_i$, $\mathbb{I}^{\alpha}_{a_i+} f(\mathbf{x})$ *and* $\mathbb{D}^{\alpha}_{a_i+} f(\mathbf{x})$ *can be expressed as*

$$\mathbb{I}^{\alpha}_{a_i+} f(\mathbf{x}) = I^{\alpha}_{\gamma+} f(\mathbf{x}) \quad \text{and} \quad \mathbb{D}^{\alpha}_{a_i+} f(\mathbf{x}) = D^{\alpha}_{\gamma+} f(\mathbf{x})(\hat{\mathbf{e}}_i). \tag{46}$$

3 Conclusions

Although Fractional Calculus has undergone an extensive study and development since its beginning, the formulation of a comprehensive theory of Fractional Differential Geometry remains as an open challenge, crucial for enhancing our understanding of Fractional Calculus. With this goal in mind, we revisited fundamental concepts related to Riemann-Liouville fractional operators and, drawing from the classical idea of path integration for scalar fields, we extended the Riemann-Liouville fractional integration and differentiation to multivariable scalar functions. As a fruitful result, a fractional generalization of a total differential was obtained. Additionally, we explored the core properties of these generalized operators and recovered various Riemann-Liouville-type definitions from existing literature, thereby contributing to the broader understanding and potential application of Fractional Calculus in multivariable contexts. A deeper analysis of the proposed operators is to be done in future contributions.

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5 Conflict of Interest

The authors declare no conflicts of interest.

6 Artificial Intelligence Usage Statement

The authors declare that they have not used any generative artificial intelligence applications, software or websites for the writing of the present paper, the design of tables and figures or the analysis and interpretation of the data.

7 Graphical Abstract

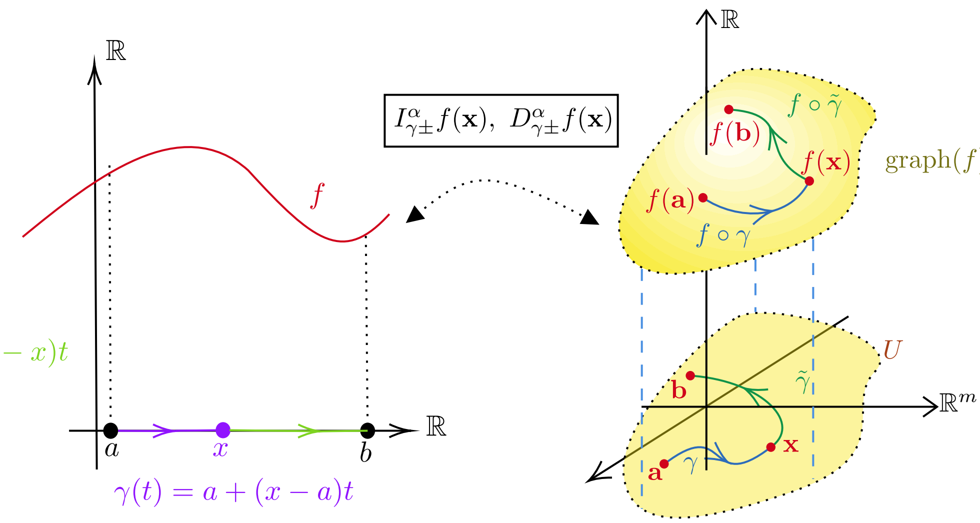

Figure 6: Graphical abstract for the present work.

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